Payoff-Relevance*

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March 25, 2012

Abstract

Markov Perfect Equilibrium as commonly used is not a solution concept defined solely on the extensive form of a game, but makes use of an externally given set of states. Using the idea that states should be payoff-relevant and that payoff-relevance should be definable from the extensive form, Maskin and Tirole have provided a canonical definition of the payoff relevant states for dynamic games with observable actions.

It is argued in this paper that their approach has to be extended in order to identify future strategic possibilities. Otherwise, different states may only differ in payoff-irrelevant ways. For a large class of games, one can apply their approach directly after modifying the extensive form of a game in payoff-irrelevant ways.

A byproduct of the analysis is that the restriction to finite action spaces and nonstationary equilibria employed by Maskin and Tirole is not necessary.

Keywords: Markov Perfect, Dynamic Game, Refinement
JEL Classification: C72, C73

1 Introduction

We prefer simple models to complicated ones. In game theory, we deal with models in which the players possess models themselves. A player has to

*Thanks to Guilherme Carmona, Venkataraman Bhaskar, Ehud Lehrer, Konrad Podczeck, Philip Reny, and Ariel Rubinstein for helpful discussions. Klaus Ritzberger has provided an extensive list of valuable comments and suggestions which are gratefully acknowledged. Christina Pawlowitsch helped me in improving the exposition. The usual disclaimer applies.

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have a model of the strategic environment she faces and that model has to contain everything that is relevant to her decision making. Yet what is relevant to her, will often depend on the models used by other players. A seemingly irrelevant variable may become important to her because other players make their behavior conditional on that variable and their behavior matters.

The model of the strategic environment used when making a choice will usually depend on past play. Partially because past play influences what players can do and want, partially because other players take history into account themselves. We want to focus on the former aspect, so we have to look at minimal models of the past and we have to look at the models of players jointly. The models players will use in this paper are simply sets of states, with each state summarizing the part of the past that is still relevant to todays decisionmaking, states are what is payoff-relevant. Players employ their models when making decisions, they choose strategies that depend on states alone. We will look at how small such models can be in order to drop as much payoff-irrelevant knowledge of the past as possible.

Our starting point is the concept of a (stationary) Markov perfect equilibrium, as defined by Maskin and Tirole in [9].[1] When payoff-relevant states were still just a modelling choice made on a case by case basis, they undertook the important task of defining the payoff relevant states in terms of the extensive form alone. In the setting of dynamic games with observable actions, they defined a Markov perfect equilibrium as a subgame-perfect equilibrium in which every players strategy is "measurable with respect to the coarsest partition of histories for which, if all other players use measurable strategies, each player's decision-problem is also measurable."

In order to show that such a partition actually exists, Maskin and Tirole had to make two restrictive assumptions: They had to assume that all players can condition on calendar time and that players can choose only among finitely many actions when called to make a decision. The first assumption goes against the spirit of payoff-relevance, the second assumption rules out many important applications of dynamic games. Both restrictions turn out to be superflous (Theorem 1).

The definition of payoff-relevant states by Maskin and Tirole is quite sensitive to what they view as decision problems. Two actions are treated differently, if they have different names. For this reason, in a Markov perfect equilibrium, players may make non-isomorphic choices in isomorphic subgames. But names of actions are not payoff relevant, they merely decorate

[1] An accessible textbook treatment of their approach can be found in [8], section 5.6.

the game tree. In general, their notion of Markov Perfect Equilibrium may tell us very little about what the payoff-relevant states are. Actually, any subgame-perfect equilibrium becomes a Markov perfect equilibrium after a simple relabeling of actions (Theorem 2). This is not just troubling on a conceptual basis, it also renders the concept inapplicable when one uses a definition of the extensive form that does not identify different actions by names. In the classical graph-theoretic definition of the extensive form of Kuhn ([6]), Markov Perfect Equilibrium cannot even be defined.

For a large class of dynamic games that are "sufficiently asymmetric", there exists a canonical labeling of actions that ensures that all Markov perfect equilibria induce isomorphic choices in isomorphic subgames (Theorem 3). Moreover, this labeling uses only information included in every reasonable definition of the extensive form (payoffs and the order structure of moves). For games outside this class of games, a relabeling of this kind may not exist.

Major proofs are collected in an appendix.

2 Environment

We will focus exclusively on games with observable actions[2] in discrete time. In the usual definition of an extensive form game, dynamics are represented by a graph and information and choices are modelled by partitions. For games with observable actions, such a definition is unnecessarily cumbersome. Information sets coincide with histories and making an arbitrary temporal ordering of moves when players move simultaneously adds only irrelevant information. For our analysis, we will take histories, which are nothing but sequences of action profiles, as the basic unit of analysis. Our formal model has three ingredients: A set of players, a set of histories and for each player, preferences over maximal histories.

We start with a set of *players* I. We merely require I to be nonempty, we do not rule out an infinity of players. Our second ingredient is a set $\mathcal{H}$ of *histories*. The members of $\mathcal{H}$ are finite or infinite sequences of I-tuples.[3] Each term of the sequence is an *action profile* and each coordinate in an action profile is an *action*. We have to impose some consistency requirements on $\mathcal{H}$:

(i) The empty sequence $\emptyset$ belongs to $\mathcal{H}$.

[2] Such games are also known as games of almost perfect information, games with perfect monitoring, or as simultaneous move games.

[3] Formally, an I-tuple is simply a function with domain I.

(ii) If $(a^t)_{t=1}^{T_1} \in \mathcal{H}$ (with $T_1 \in \mathbb{N} \cup \{\infty\}$) and $T_2 < T_1$ then $(a^t)_{t=1}^{T_2} \in \mathcal{H}$.

(iii) If $(a^t)_{t=1}^{T} \in \mathcal{H}$ for all $T \in \mathbb{N}$ then $(a^t)_{t=1}^{\infty} \in \mathcal{H}$.

(iv) Suppose h is not of maximal length. For each player i, let $A_i(h) = \{a_i : \exists a_{-i}(h, (a_i, a_{-i})) \in \mathcal{H}\}$. Then for all $a \in \prod_{i \in I} A_i(h)$, $(h, a) \in \mathcal{H}$.

The conditions (i) and (ii) ensure that $\mathcal{H}$ has a well defined tree structure. Condition (iii) for infinite histories ensures that maximal histories exist even when the time horizon is infinite. Finally, condition (iv) guarantees that the actions a player can play after a certain history, are independent of what the others play after that history.

We call histories of maximal length *plays* and all other histories *nonterminal histories*. We denote the set of nonterminal histories by $\mathcal{H}^*$, these are the histories where players can actually act. Clearly, they all have finite length. The last ingredient in our definition of the extensive form is a reflexive, complete and transitive preference ordering $\preceq_i$ over plays for each player i. So $(I, \mathcal{H}, (\preceq_i)_{i \in I})$ is a *game*. We may refer to $\mathcal{H}$ also as a *game form*.

This framework is quite flexible. We can model situations in which only some players can choose after some history by allowing the other players only to "choose" one single action. This way, we can accommodate games of perfect information, games with overlapping generations of players as in [3], and games with long-and short run players as in [4]. Technically, we could even work with histories indexed by large ordinals and accommodate transfinite games such as long cheap talk ([2]).

A (pure) *strategy* for player i is a function s_i that maps each nonterminal history h to an action in $A_i(h)$. We restrict ourselves to pure strategies in order to avoid measurability problems when dealing with continua of actions.

When we specify a strategy for each player, we get a *strategy profile* $(s_i)_{i \in I}$. By recursion, every strategy profile determines a unique play. A *continuation* c of a nonterminal history h is a sequence of action profiles that makes the sequence (h, c) a play.[4] So a continuation is "what can happen after h". At each nonterminal history, every player i has conditional preferences $\preceq_i^h$ over continuations such that $c \preceq_i^h c'$ if and only if $(h, c) \preceq_i (h, c')$.

[4]The notation (h, c) denotes the concatenation of the sequences h and c. The context should prevent confusion when the same notation is used for ordered pairs. What we call continuations, are *futures* in [9].

Given a nonterminal history h of length T, we can construct an h-*subgame* $(I, \mathcal{H}^h, (\preceq_i^h)_{i \in I})$ in a natural way. The set of histories consists of all continuations and initial segments of continuations. The continuations are the plays and preferences for player i are simply $\preceq_i^h$. Every strategy s_i for player i induces a uniques *continuation strategy* $s_i|h$ in the h-subgame by setting $s_i|h(h') = s_i\big((h, h')\big)$. Since every strategy profile induces a unique play, every player has preferences over her strategies, for fixed strategies of the others. By slight abuse of notation, we also write $s_i \preceq_i s$. A *subgame-perfect equilibrium* of the game $(I, \mathcal{H}, (\preceq_i)_{i \in I})$ is a strategy profile such that for every player i, every nonterminal history h, and every possible strategy s of player i, the inequality $s|h \preceq_i^h s_i|h$ holds.

Finally, we will need some mathematical definitions related to partitions: A partition Π of a set X is a family of nonempty, pairwise disjoint subsets of X such that $\bigcup_{P \in \Pi} P = X$. We call the elements of a partition *cells*. If Π_1 and Π_2 are partitions of X, we say that Π_1 is *coarser* than Π_2 if every cell in Π_1 is the union of cells in Π_2. This is equivalent to every cell in Π_2 being the subset of a cell in Π_1. Let Π be a partition of X and f be a function defined on X. We say that f is Π-*measurable*, if for every cell $P \in \Pi$ and every two elements $x, y \in P$, we have $f(x) = f(y)$. When x and y are in the same cell of the partition Π, we will say that x and y are Π-*equivalent*.

3 Markov Perfect Equilibrium

A *partition profile* $(\Pi_i)_{i \in I}$ lists for each player i, a partition Π_i of $\mathcal{H}^*$. A partition profile should provide a model of the relevant past for every player that encompasses the current strategic environment. For this, we need the following consistency conditions, taken from [9]. A partition profile $(\Pi_i)_{i \in I}$ is *consistent* if it satisfies the following two conditions:

(i) If the histories h_1 and h_2 are Π_i-equivalent for any player i, then h_1 and h_2 have the same continuations.

(ii) Suppose all players $j \neq i$ employ Π_j-measurable strategies. If h_1 and h_2 are Π_i-equivalent, then $s_i|h_1 \preceq_i^{h_1} s_i'|h_1$ if and only if $s_i|h_2 \preceq_i^{h_2} s_i'|h_2$ for all strategies s_i and s_i' of player i such that $s_i|h_1 = s_i|h_2$ and $s_i'|h_1 = s_i'|h_2$.

Condition (i) formalizes the idea that all players can behave in the same way in the future and condition (ii) that one can restrict oneself to measurable strategies, provided everybode else does the same. Condition (ii) does not imply that the preferences over continuations are the same. Consider two histories ending in the following one-shot games:

	L	R
T	(3,3)	(1,3)
B	(3,1)	(1,1)

	L	R
T	(1,1)	(3,1)
B	(1,3)	(3,3)

Clearly, both players have different preferences over continuations. But due to the special structure of this example, a players payoff depend in both games only on the action chosen by the other players. So both players are indifferent between their actions and hence continuation strategies. So (ii) holds trivially.

A partition profile $(\Pi_i)_{i\in I}$ is *dated* if any two histories h_1 and h_2 that are Π_i-equivalent for some player i are of the same length. In a dated partition profile, measurable strategies can be condition on calendar time.

What we are really after are minimal models. We say that the partition profile $(\Pi_i)_{i\in I}$ is *coarser* than the partition profile $(\Pi_i')_{i\in I}$ if Π_i is coarser than Π_i' for every player i. A coarser partition profile corresponds to players using smaller models.

Theorem 1 *There exists a coarsest consistent partition profile and a coarsest consistent dated partition profile.*

The maximally coarse consistent partition profile contains the payoff-relevant information for every player, they form the state space for the player. A special case of this result has been obtained by Maskin and Tirole. They show that a coarsest consistent dated partition profile exists when no player can choose among infinitely many actions after some history and when the set of players is finite. For finitely many players, they have also shown that maximally coarse consistent partition-profiles exist in general by emplying Zorn's lemma. They did not rule out the possibility of different, incomparable, maximally coarse consistent partitions in that case.

In applied work, one usually assumes that there is a single state spaces shared by all players. Our formulation allows for different players to have different partitions in the coarsest consistent partition profile. One could easily formulate everything in terms of a partition common to all players. It is also possible to rule out a diversity of partitions by essentially requiring that everything some player does has an impact on every other player at every time. This is the approach employed by Maskin and Tirole. The assumption is satisfied in games in which everyones strategic possibilities depend on some aggregate stock, such as the stock of fish in the classical fishwar model ([7]).

Finally, Maskin and Tirole define a *Markov Perfect Equilibrium (MPE)* as a subgame perfect strategy profile in which every player plays a strategy measurable with respect to her partition in the coarsest consistent dated partition profile. They define *stationary Markov Perfect Equilibrium (SMPE)* as a subgame perfect strategy profile in which every player plays a strategy measurable with respect to her partition in the coarsest consistent partition profile. For expositional clarity, we focus on the latter. The discussion can easily be adopted to the dated case.

4 Limitations of MPE and SMPE

When we defined consistent partition profiles, we required that two histories are equivalent for any player only if they allow for the same continuations. But sequences of action profiles are a poor representation of a decision problem. In a (S)MPE, players may face isomorphic subgames differing only by the names of actions and behave differently in the two subgames. So (S)MPE does not conform to reasonable notions of subgame consistency. To make this precice, we need to formalize a notion of being essentially the same.

An *isomorphism* between the games $(I, \mathcal{H}, (\preceq_i)_{i\in I})$ and $(I, \mathcal{H}', (\preceq_i')_{i\in I})$ is a bijection $f : \mathcal{H} \to \mathcal{H}'$ such that:

(i) For two histories h_1 and h_2, h_1 is an initial segment of h_2 if and only if $f(h_1)$ is an initial segment of $f(h_2)$.

(ii) For any nonterminal history h there exists a family $(f_i^h)_{i\in I}$ of bijections $f_i : A_i(h) \to A_i'(f(h))$ such that $f\big(h, (a_i)_{i\in I}\big) = \big(f(h), (f_i^h(a_i))_{i\in I}\big)$.

(iii) For any two plays h_1 and h_2 and any player i, $h_1 \preceq_i h_2$ if and only if $f(h_1) \preceq_i' f(h_2)$.

Condition (i) guarantees that f is an order-isomorphism under the "is an initial segment of"-ordering. Condition (ii) makes it an order isomorphism on the terminal histories for the preferences of each player. Condition (ii) is required to preserve the internal product structure of histories.

Isomorphic games are the same in everything that is payoff relevant. But by a mere relabeling of actions, we can change every subgame-perfect equilibrium into a stationary Markov perfect equilibrium.

Theorem 2 *Given any game, there exists an isomorphic game in which every subgame-perfect equilibrium is a stationary Markov perfect equilibrium.*

If f is an isomorphism between the games $(I, \mathcal{H}, (\preceq_i)_{i\in I})$ and $(I, \mathcal{H}', (\preceq_i'$ $)_{i\in I})$ and $(s_i)_{i\in I}$ is a strategy profile for $(I, \mathcal{H}, (\preceq_i)_{i\in I})$, there is also an *induced strategy profile* $(s_i^f)_{i\in I}$ for $(I, \mathcal{H}', (\preceq_i')_{i\in I})$. Let h be a nonterminal history in $\mathcal{H}$ of length T and h' be the unique history of lenth $T+1$ that occurs when all players follow the strategy profile $(s_i)_{i\in I}$. Now $s_i^f\big(f(h)\big)$ is simply defined to be the i^{th} coordinate of the last term of $f(h')$.

With this definition out of the way, we can define subgame consistency. A strategy profile $(s_i)_{i\in I}$ in a game $(I, \mathcal{H}, (\preceq_i)_{i\in I})$ is *subgame-consistent* if, whenever there is an isomorphism f between two subgames, the strategy profile induced by f in one subgame is the strategy profile in the other subgame. Informally, in a subgame-consistent strategy profile, players facing the same strategic situation behave in the same way.[5] By our last result, there clearly exists stationary Markov perfect equilibria that are not subgame-consistent.

Suppose there are two different isomorphisms between subgames. Then some strategy profile in one subgame induces different strategy profiles under the different isomorphisms. Such a strategy profile cannot be subgame-consistent. To avoid this problem, we look at games in which two subgames can be isomorphic in only one way. Formally, we say that a game is *rigid* if the only isomorphism of the game with itself is the identity mapping.

Remark 1 *A game is rigid if and only if no two subgames are isomorphic to each other in more than one way.*

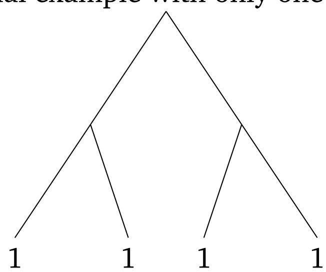
Proof: Since the whole game is a subgame, if two subgames are isomorphic to each other in one way, the game must be rigid. Conversely, we observe first that every isomorphism from a subgame to itself can be extended to an isomorphism on the whole game that maps each history outside the subgame to itself. So in a rigid game, every subgame can be isomorphic to itself only under the identity. Now, if f and f' are different isomorphism from one subgame to another, then $f^{-1} \circ f$ and $f^{-1} \circ f'$ are different isomorphisms of the first subgame to itself, whic cannot be in a rigid game. ■

For rigid games, we can relabel actions in a way that makes stationary Markov perfect equilibria subgame-consistent.

[5]The notion of subgame-consistency goes back to [12].

Theorem 3 *Given any rigid game, there exists an isomorphic game such that every stationary Markov perfect equilibrium is subgame-consistent.*

For games that are not rigid, a mere relabeling will not suffice. Here is a trivial example with only one player:



The two proper subgames are clearly isomorphic, but no pure strategy can make all isomorphic choices at the same time. Any isomorphism reproduces the tree structure, including the payoff-irrelevant redundancies. There is really no decision problem in this example. The example is nongeneric, but this is natural in this context. The symmetries avoided by rigid games stem from nongeneric payoff-ties.

Remark 2 *If in game with only finitely many histories there exists a player who is not indifferent between any two plays, then the game is rigid.*

Proof: By assumption, there is a player i whose preferences $\preceq_i$ over plays are a linear ordering. Isomorphisms are completely determined by what they do to plays. The identity on $\mathcal{H}$ is the only isomorphism to itsef that doesn't change any plays. Now by (iii) in the definition of an isomorphism, the restriction of every isomorphism to the set of plays is an order-isomorphism for $\preceq_i$. The only order isomorphism of a finite, linearly ordered set to itself is the identity. ■

5 Discussion

The notion of subgame consistency used here is quite strong. In particular, we assume that *all* isomorphisms induce behavior in corresponding subgames. This corresponds to the idea that an observer can hold any view on how different subgames are related to each other. A different approach would require that if two subgames are related by *some* isomorphism, there exists *some* isomorphism that relates the behavior in one subgame to the behavior in another subgame. For elementary games, these notions coincide.

For other games, requiring this form of subgame-consistency would mean that players have a shared understanding of what subgames are considered to be the same. This requires them to have a more complex model of their environment that is not just based on the structure of the game.

Satisfying our notion of subgame-consistency would be easier if players could employ mixed strategies. Our notion does rule out certain pure equilibria even in one-shot games, where symmetric equilibria always exist in mixed strategies ([10]). If one prefers to work with the mixed extension, one replaces actions by behavior strategies, interpreted as transition probabilities. To justify this, one appeals to a suitable extension of Kuhn's theorem, such as [1]. This will require additional assumptions. The extension is relatively straightforward when only finitely many players have nontrivial choices each period and after each nonterminal history, a player can play at most countably many actions. In this case, $\mathcal{H}^*$ will be at most countable and every consistent partition profile will only include partitions with at most countably many cells. No special measurable or topological structure has to be imposed on the state space. Also, if we start with a dynamic game defined in terms of a given countable state space, we are guaranteed that our construction gives us less states (Proposition 5.6.2. in [8]), so for many applications, there is no problem in admitting mixed strategies.

Finally, it is worth pointing out that one could separate all results from specific solution concepts such as SPE. Effectively, we have shown that one can represent future strategic possibilities by summary statistics, states, and that for rigid games, one can do this in a way that preserves symmetries in the strategic possibilities. If we look at purely forward-looking forms of behavior, states will contain all relevant information.

6 Appendix

The following proof builds on the original proof of Maskin and Tirole and work of Ore on the lattice structure of partitions. The construction of the coarsest consistent partition profile is based on a characterization of the finest common coarsening of a family of partitions. Partitions of a set form a complete lattice with the "coarser than"-ordering and Ore laid the foundations for the theory in [11]. The reader interested in partition lattices will find a comprehensive overview in [5], section IV.4.

Proof of Theorem 1

We show that a coarsest consistent partition profile exists. Essentially the same proof works for dated partition profiles. For each player i, we define a relation $\equiv_i$ on $\mathcal{H}^*$ such that $h \equiv_i h'$ if there is a finite sequence of nonterminal histories $h, h_1, h_2, \dots, h_n, h'$ such that consecutive histories in the sequence are equivalent for the partition of i in some consistent partition profile Π_i. The relation $\equiv_i$ is clearly an equivalence relation. Let Π_i^* be the partition of $\mathcal{H}^*$ into $\equiv_i$-equivalence classes.

We will now show that $(\Pi_i^*)_{i\in I}$ is the coarsest consistent partition profile. It is clearly a partition profile and by construction coarser than any consistent partition profile. It remains to verify that it is also consistent.We begin with consistency condition (i). If h and h' are Π_i^*-equivalent, they are also $\equiv_i$-equivalent and that means they are connected by a finite sequence of nonterminal histories such that consecutive histories are equivalent under some partition in a consistent partition profile. But then consecutive histories in the sequence must allow for the same continuations by (i) and by transitivity, h and h' allow for the same continuations. This proves (i).

For consistency condition (ii), observe that the coarser a partition is, the less strategies are measurable. That means that whenever all players $j \neq i$ play Π_j-measurable strategies, they play strategies measurable with respect to their partition in every consistent partition profile. Now suppose all players $j \neq i$ employ Π_j^*-measurable strategies. Let h and h' be Π_i^*-equivalent and $s_i|h \preceq_i^h s_i'|h$ with $s_i|h_1 = s_i|h_2$ and $s_i'|h_1 = s_i'|h_2$. Then there exists a finite sequence $h_1, \dots, h_{n+1}$ of nonterminal histories and a finite sequence of consistent partition profiles $(\Pi_i^1)_{i\in I}, \dots, (\Pi_i^n)_{i\in I}$ such that $h_1 = h$, $h_{n+1} = h'$ and h_k, h_{k+1} are in the same cell of Π_i^k for $k = 1, \dots, n$. Now if all players $j \neq i$ employ Π_j^*-measurable strategies, they also employ Π_j^k-measurable strategies. Since all continuations are the same, we can modify s_i and s_i' such that $s_i|h^k = s_i|h$ and $s_i'|h^k = s_i'|h$ for $k = 1, \dots, n$. Hence, $s_i|h^k \preceq_i^{h_k} s_i'|h^k$ for $k = 1, \dots, n$. By transitivity of $\equiv_i$, consistency condition (ii) holds. ■

Proof of Theorem 2

Proof: Let $\big(\mathcal{H}, (\succeq_i)_{i\in I}\big)$ be a given game. We rename every action played at a certain history so that actions played after different histories get different names. Here is one way to do this: If $a = (a_i)_{i\in I}$ is an action profile and z some mathematical object, write $a|z$ for $\big((a_i, z)\big)_{i\in I}$. Set $f(\emptyset) = \emptyset$. Now suppose $f(h)$ is already defined when h has length $T-1$. For a finite history

$h = (a^t)_{t=1}^{T}$ of length T, set

$$f(h) = \Big(f\big((a^t)_{t=1}^{T-1}\big), a^T|h\Big).$$

If $h = (a^t)_{t=1}^{\infty}$ is an infinite history, let $f(h)$ be the unique sequence such that the initial segment of length T coincides with $f\big((a^t)_{t=1}^{T}\big)$ for all T. Set $\mathcal{H}' = f(\mathcal{H})$, which is a game form. Define $(\preceq_i')_{i\in I}$ on elements of $f(\mathcal{H})$ of maximal length by $f(h) \preceq_i' f(h')$ if and only if $h \preceq_i h'$ for every player i. It is obvious that f is an isomorphism.

Since actions played after different histories are different, there are no two histories in $\mathcal{H}'$ that have a common continuation. So the only stationary consistent partition consists of all singletons and every strategy is measurable with respect to this partition. So every subgame perfect equilibrium uses strategies measurable with respect to this partition and is therefore a SMPE. ■

Proof of Theorem 3

Proof: Let $G = (I, \mathcal{H}, (\preceq_i)_{i\in I})$ be a rigid game. By transforming the game as in the proof of Theorem 2, we can assume that no action can be played after different histories. For each player $i \in I$ we define a function g_i on $A_i = \bigcup_{h\in\mathcal{H}^*} A_i(h)$. By assumption, for each $a_i \in A_i$ there is a unique history h_{a_i} such that a_i can be played after h_{a_i}. Let (a_i, a_{-i}) be a profile with $\big(h_{a_i}, (a_i, a_{-i})\big) \in \mathcal{H}$. Now we let $g_i(a_i)$ be the set of all $a_i' \in A_i$ such that there is some h-subgame isomorphic to the h_{a_i}-subgame with isomorphism $f : \mathcal{H}^{h_{a_i}} \to \mathcal{H}^{h}$ such that a_i' is the i^{th} coordinate of the last component of $f\big(h, (a_i, a_{-i})\big)$. Effectively, $g_i(a_i)$ is the set of actions that serve the same role as a_i in some subgame.

We now construct a function g on $\mathcal{H}$ recursively. We set $g(\emptyset) = \emptyset$. Suppose g is already defined for all histories of length $n-1$ and $h' = \big(h, (a_i)_{i\in I}\big)$ is a history of length n. Then we set $g(h') = \big(g(h), (g_i(a_i))_{i\in I}\big)$. If $h = (a^t)_{t=1}^{\infty}$ is an infinite history, let $g(h)$ be the unique sequence such that the initial segment of length T coincides with $g\big((a^t)_{t=1}^{T}\big)$ for all T. Clearly, g is an injection.

Now $g(\mathcal{H})$ is a game form. Define $(\preceq_i')_{i\in I}$ on elements of $g(\mathcal{H})$ of maximal length by $g(h) \preceq_i' g(h')$ if and only if $h \preceq_i h'$ for every player i. Then $G' = (I, g(\mathcal{H}), (\preceq_i')_{i\in I})$ is a game isomorphic to G under the isomorphism $g : \mathcal{H} \to g(\mathcal{H})$. Moreover, any isomorphic subgames of G' actually coincide by construction, so every SMPE of G' is necessarily subgame-consistent. ■

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