Econ Theory (2015) 58:125–160
DOI 10.1007/s00199-013-0778-7

RESEARCH ARTICLE

Collusive communication schemes in a first-price auction

Helmuts Āzacis · Péter Vida

Received: 24 August 2012 / Accepted: 24 September 2013 / Published online: 18 October 2013

Abstract We study optimal bidder collusion in an independent private value first-price auction with two bidders and two possible valuations. There is a benevolent center that knows the bidders' valuations and sends private signals to the bidders in order to maximize their expected payoffs. After receiving their signals, bidders compete in a standard first-price auction, that is, without side payments or bid restrictions. We find that to improve on the bidders' payoffs, the signals must depend upon the valuations. If the bidders' signals are restricted to be non-correlated (depend only on the opponent's valuation), then the bidders' payoffs are strictly higher than the larger possible set of signals. If the signals are restricted to be perfectly correlated (public), only two possible signals are needed to achieve the highest bidder payoffs. However, these payoffs can be improved upon if the two signals are allowed to be imperfectly correlated.

Keywords Bidder-optimal signal structure · Bid coordination mechanism · Collusion · (Bayes) correlated equilibrium · First-price auction · Public and private signals

We would like to thank Roberto Burguet, Amrita Dhillon, Françoise Forges, Bernard Lebrun, Volcker Nocke, Martin Peitz, Nicolas Schutz, the editor, and two anonymous referees for their comments that have helped to improve this paper. Péter Vida acknowledges the financial support from the Graf Hardegg'sche Stiftung.

H. Āzacis (✉)
Cardiff Business School, Cardiff University, Aberconway Building,
Colum Drive, Cardiff CF10 3EU, Wales, UK
e-mail: azacish@cf.ac.uk

P. Vida
Department of Economics, University of Mannheim, L7, 3-5, 68131 Mannheim, Germany
e-mail: vidapet@gmail.com

JEL Classification D44 · D82

1 Introduction

In a standard independent private value (IPV) first-price auction, it is assumed that each bidder observes her own valuation but has no information about her opponents' valuations, except for the distribution from which they are drawn. However, in real-life auctions, bidders may hold or may have incentives to gather additional information about their opponents' valuations. For example, in procurement auctions, while a firm's actual cost is private information, other firms may observe its investment in new technologies, which serves as a noisy signal about the firm's actual cost. Fang and Morris (2006), Bergemann and Vällimäki (2006), and Kim and Che (2004) are examples of setups where bidders can benefit from such noisy signals, and the seller expects less revenue in first-price auctions compared with second-price auctions.[1]

Bidders can also improve their position against the seller by acquiring information about each other's private information through various institutions. It is well recognized that trade associations, through gathering and sharing aggregated industry-specific information, can serve as collusion-facilitating devices.[2] For example, Genesove and Mullin (1997) provide an interesting case study of the workings of the Sugar Institute, the trade association uniting the US domestic sugar refiners from 1928 to 1936. They describe how the Sugar Institute collected, aggregated, and disseminated the data about the industry among its members. One of their findings is that "the Sugar Institute revealed less information to its members than it knew" (Genesove and Mullin 1997, p. 20). Indeed, our results will also imply that bidders are better off if the signals observed about their opponents' valuations contain some noise instead of being perfect signals. This is because, in the latter case, bidders would bid in a complete information environment.

Other institutional arrangements that facilitate information exchange include forming a joint venture or contracting a consultancy company. Thus, for example, the Italian competition authority found several oil companies guilty of collusion in 2006.[3] These companies competed in tenders for the right to supply fuel to airline companies, but at the same time, they also had joint ventures for storage and onboard refueling services at individual airports. The competition authority concluded that the joint ventures were used as a platform for information exchanges between the oil companies to coordinate their bids for fuel supply contracts. In the Organic Peroxides case,[4] Swiss consultancy AC Treuhand played an instrumental role in the information exchange between cartel members. Among other things, the consultancy organized meetings, collected data,

[1] Information about the opponents' valuations has no impact on the equilibrium bidding strategies in second-price private value auctions and hence does not affect the revenue of the seller.

[2] See, for example, Vives (1990) and the references in Sect. 2 of that paper.

[3] Case I641, Rifornimenti Aeroportuali, Italian Competition Authority Decision No. 15604. For a critical discussion of this case, see Caffarra and Kühn (2006).

[4] Case COMP/E-2/37.857—Organic Peroxides, European Commission decision. To learn more about this case, see, for example, Marshall and Marx (2012).

organized the auditing of these data, and provided the members with the relevant statistics. Finally, bidders occasionally hire third parties that help them to prepare bids and all other necessary documentation in public procurement auctions. This, again, creates possibilities for collusion. For example, on one occasion, the Latvian competition authority found two construction firms guilty of information exchange by employing the same third party to prepare their offers.[5]

Hence, for collusive purposes, bidders may agree and commit to mechanisms that provide them with (noisy) information about each other's valuations. Our goal is to examine the extent to which bidders can improve their payoffs in a one-shot IPV first-price auction when they have access to such information sharing mechanisms. In particular, we assume that the bidders have access to a center, an incentiveless third party, which facilitates collusion between them. For the collusion to be successful, the center must be able to learn the private valuations of bidders and then, conditional on this information, coordinate the bids that the bidders submit in the auction. In this paper, we abstract away from the process by which the center collects information about bidders' valuations. Instead, we assume that the center already knows the realized valuation profile.[6] The center, after having observed the realized valuation profile but before the bidding takes place, sends noisy (random) private signals to the bidders about each other's valuations. As in a standard IPV setup, each bidder knows her own valuation as well. Note that a signal does not alter a bidder's valuation of the object, but rather it changes the bidder's beliefs about the opponent's valuation. Since the signals can be correlated, a signal can also convey information about the signal received by the opponent. After learning their valuations and the signals sent by the center, the bidders are allowed to bid as they want. However, by altering bidders' beliefs about their opponents' valuations, the center affects the equilibrium strategies of bidders in the first-price auction. The question we seek to answer is: *what is the signal structure that the center should use to maximize bidders' payoffs?*[7]

As in Fang and Morris (2006), we consider the simplest IPV setup with only two bidders, whose valuations can take either low or high value. Fang and Morris (2006) show that the seller expects less revenue in the first-price auction than in the second-

[5] Case p/11/03.01.-01./8, Competition Council of Latvia Decision No. E02-66.

[6] Forges (2006) names such a center as an omniscient mediator and justifies it on several grounds. Given our earlier examples, we can assume that the role of center is played by a trade association or a consultancy company that has the right to audit accounting books of the bidders or that the center is a president of joint venture who has the access to confidential information of the bidders that own that joint venture. Our analysis remains valid if we additionally assume that the bidders pay some fixed amount to the center. To justify further the existence of the omniscient center, we show in our working paper version (Āzacis and Vida 2012) how the center can elicit the information about the bidders' valuations in a weakly incentive compatible manner. We assume that if a bidder reports some valuation to the center, then she is not allowed to bid above it in the auction.

[7] Different correlated equilibrium concepts, when applied to games with incomplete information, are extensively studied in Forges (1993, 2006) and Bergemann and Morris (2011). According to the equilibrium classification of Forges (1993, 2006), we look for the best (from the bidders' point of view) *Bayesian solution* of a *standard* IPV first-price auction, where bidders can make use of an omniscient mediator. According to Bergemann and Morris (2011), we look for the best *Bayes correlated equilibrium* of the standard IPV first-price auction. These two concepts coincide in the current setup; see the discussion in Section 4.1 of Bergemann and Morris (2011).

price auction if (a) signals are drawn *independently* (that is, a signal only depends on the valuation of the opponent), (b) the signal received by a bidder is her *private* information, and (c) the signals can take one of *two* values. We generalize the signal structure in several directions and address the following questions: Can the center increase bidders' payoffs further by using correlated signals? Should the signals be distributed publicly or privately? Are the bidders strictly better off if the center uses richer language instead of two-valued signals?

We analyze and compare four classes of signal structures. First, we keep the two-valued signals as in Fang and Morris (2006), but we allow them to be correlated. We characterize the unique symmetric equilibrium for any two-valued symmetric signal structures and then find the signal structure that maximizes the bidders' joint payoff. Under the optimal symmetric signal structure, the signals received by the bidders are neither independent nor perfectly correlated. Roughly speaking, a high-valuation bidder, depending on the signal she receives, learns either her opponent's valuation or the signal that the opponent has received, but not both. Therefore, it is possible that both bidders have high valuations, and both know that the opponent has a high valuation, but each is still unsure as to what the opponent knows. (Since it is also possible that one's opponent may be uncertain whether one's valuation is high or low.) As a result, in this case (and overall), the bidders bid less aggressively.

An important subclass of the two-valued symmetric signal structures contains those in which the randomness does not depend on the bidders' realized valuation profile. In order to produce such signals, the center does not have to know the profile of bidders' valuations. Hence, the question: Can the center induce collusive bidding without knowing the bidders' valuations, and simply distribute correlated signals? We find that the answer is negative. To be more precise, an equilibrium of the extended game in which the bidders receive valuation-independent correlated signals, and then they bid, is known as a strategy-correlated equilibrium of the underlying first-price auction.[8] We find the following as a by-product of the analysis of two-valued (valuation-dependent) symmetric signal structures. No collusive equilibrium in symmetric correlated strategies exists if the center uses only two-valued valuation-independent correlated signals. More precisely, in any symmetric strategy-correlated equilibrium, the bidders bid exactly as in the non-cooperative equilibrium, independently of the signals received. This result suggests that for the collusion to be profitable, the center must possess some information about the valuations.

What happens when the signal structure is richer? Although the analysis of the general correlated signal structure, when the signals are allowed to take more than two values, was too complicated,[9] we were able to consider two extreme cases of n-valued signal structures, when the signals are either independent or perfectly correlated. When the signals are perfectly correlated, the center, depending on the valuation profile of the bidders, effectively chooses a *single* signal out of n possible values and announces it publicly. We establish that within the class of public signals, the bidders can achieve

[8] We refer here to Aumann's (1974) strategic-form correlated equilibrium, which has been applied to games with incomplete information by Cotter (1991) and Forges (1986, 1993). See also the references in footnote 7.

[9] However, we conjecture that introducing more signal values does not increase the bidders' payoffs, that is, the optimal two-valued correlated signal structure gives the overall optimum.

their best payoffs with a two-valued signal, and allowing richer n-valued public signal does not improve bidders' payoffs. Moreover, we show that there is no gain in having an asymmetric public signal structure since the optimum can be implemented with a symmetric one. This implies that, in terms of payoffs, any public signal structure is dominated by the optimal two-valued correlated signal structure discussed previously. However, the optimal public signal structure has the attractive property that, unlike the case of the optimal two-valued correlated signal structure, it is independent of the prior beliefs of bidders. Therefore, the center might prefer to use the optimal public signal if he is unsure of the prior beliefs of bidders.[10]

The result that a two-valued public signal maximizes bidders' payoffs within the class of public signals is in sharp contrast with our findings for the class of signal structures, where the signals are drawn independently and identically. We show that for any n-valued independent signal structure, we can always construct an $(n + 1)$-valued independent signal structure that results in strictly higher payoffs for the bidders.

Compared to the bidders' payoffs under the optimal public signal structure, we find numerically that the optimal n-valued independent signal structure gives higher payoffs only when the prior probability of a bidder having a low valuation is low. Otherwise, the optimal public signal structure leads to higher payoffs. Numerical results also indicate that the bidders' payoffs under the optimal two-valued correlated signal structure always exceed those that result from any independent signal structure for any n and any prior beliefs.

We should emphasize that, except for the public signal case, we have restricted attention to symmetric signal structures. Apart from tractability issues that arise when dealing with asymmetric signal structures, this decision was motivated by a result in Fang and Morris (2006). They show in their Proposition 5 that even in the two-valued independent signal case, neither symmetric nor asymmetric equilibrium exists for generic values of parameters once we consider asymmetric signal structures. For all symmetric signal structures that we study, we find a unique symmetric Nash equilibrium; therefore, the existence of equilibrium is not an issue in our model.

1.1 Related literature

McAfee and McMillan (1992) characterize optimal collusive mechanisms for bidders when the center needs to elicit their private valuations in an incentive compatible manner.[11] However, they assume that the center can enforce the bids. On the contrary, we assume that the center knows the valuation profile but cannot enforce

[10] Public signals can also arise in the absence of the center. For instance, if the bidders participate in a sequential auction for multiple objects and their valuations are correlated across the objects, then any public information about outcomes of earlier auction rounds will serve as a public signal about bidders' valuations in later auction rounds. This, for example, happens in Ding et al. (2010) and Āzacis (2013).

[11] Several recent papers—Chakravarty and Kaplan (2013), Condorelli (2012), Hartline and Roughgarden (2008), Yoon (2011)—build on McAfee and McMillan (1992) and study mechanisms that maximize agents' joint surplus when side payments are not allowed. However, using the terminology of Marshall and Marx (2007), all these mechanisms are bid submission mechanisms, while our mechanism is a version of bid coordination mechanism.

the bids. Marshall and Marx (2007) and Lopomo et al. (2011) extend the model of McAfee and McMillan (1992) by assuming that the center cannot control the bids that the bidders submit at the auction, but he can enforce side payments between the bidders. In particular, Lopomo et al. (2011) show that in this case, there is no collusive mechanism that improves bidders' payoffs relative to non-cooperative bidding even if side payments, which only depend on the reported valuations, are allowed. Despite this negative result, we believe it is still of interest to study how the center should share his knowledge about bidders' valuations if he possessed such information, and if he could not control the bids directly or indirectly through monetary transfers.[12]

There exists extensive literature that studies a similar question but from the seller's perspective. Namely, how should the seller disclose information about bidders' valuations in order to maximize his revenue? This question has been studied both when the auction rules are fixed and in the mechanism design context. Among the former, closest to our setup is that of Kaplan and Zamir (2000) who also investigate a first-price private value auction.[13] They find that the seller can increase his revenue through public announcements about bidders' valuations. We show that the opposite result holds in our model (see Remark 2). These differences can be reconciled with the help of Maskin and Riley (2000a) who compare seller's revenues in first-price and second-price auctions in the presence of asymmetries: The setup in Kaplan and Zamir (2000) is closer to the one considered in Proposition 4.3 (and Example 1) of Maskin and Riley (2000a), while our setup is closer to the one in Proposition 4.5 (and Example 3). In the context of mechanism design, when the seller additionally decides on the auction format, Skreta (2011) considers a model, in which, similar to our model, the bidder's type is multidimensional consisting of valuation and belief components. Skreta (2011) finds that in the IPV setup, the maximal revenue is obtained with full information disclosure, but the optimal mechanism is different from the first-price auction.

Finally, Eliaz and Serrano (2013) and Eliaz and Forges (2012) also consider the problem of an informed center who decides at each state of nature what information to disclose to two agents who interact in a strategic-form game, but instead of the first-price auction, Eliaz and Serrano (2013) analyze a multiaction prisoners' dilemma, while Eliaz and Forges (2012) study a class of games in which payoffs are quadratic and the agents' actions are substitutes.

The rest of the paper is organized as follows. In Sect. 2, we set up the model. Next, we study the four classes of signal structures in the same order as discussed above. The general two-valued signal structures, including the correlated signals that do not depend on the bidders' valuations, are studied in Sect. 3, the public signal structures in Sect. 4, and the independent signal structures in Sect. 5. We conclude in Sect. 6. All major proofs are relegated to the "Appendix".

[12] The presence of entry costs can also be exploited to achieve collusion; see, for example, Miralles (2010).

[13] For an example of information disclosure in an all-pay auction, see Kaplan (2012). Quint (2010) studies both information acquisition by bidders and information revelation by the seller in a first-price common value auction.

2 The model

Two bidders, 1 and 2, compete for an object. When we refer to a generic bidder, we use ‘she’ and we do not use an index to indicate the bidder unless it causes confusion. Bidders’ valuations of the object are independently drawn from identical distributions. We assume that a bidder’s valuation of the object takes one of the two possible values $\{V_L, V_H\}$, where $V_L = 0$ and $V_H = 1$.[14] The ex-ante probability that a bidder’s valuation v takes value V_L is denoted by $p \in (0, 1)$, and the probability of V_H is $1 - p$.

As in standard private value auction models, each bidder privately observes her own valuation v. Fang and Morris (2006) assume that each bidder also privately observes a noisy signal s about her opponent’s valuation. For tractability, they assume that the noisy signal can only take two possible qualitative categories, $s \in \{L, H\}$.[15] Further, in Fang and Morris (2006), the signals are drawn independently and identically, conditional on realized valuations, from a given distribution function. This implies that bidder’s signal conveys no information about the signal received by the opponent. We generalize their setup by allowing the signals to be correlated and/or to take on values from a larger set, namely, from the set $N = \{1, 2, \ldots, n\}$.

We assume that the bidders have access to an incentiveless center who knows the realized valuations of bidders and, depending on these, sends private signals to the bidders prior to the auction. The goal of the center is, through the choice of distribution function for the signals, to maximize the joint payoff of bidders, that is, the sum of bidders’ ex-ante payoffs. It is further assumed that the center cannot employ monetary transfers or impose any restrictions on bids in order to weaken the competition between the bidders.

Let $s_l \in N$ denote a private signal received by bidder l $(l = 1, 2)$. The signals are generated as follows. For all $(i, j) \in N \times N$,

$$\begin{aligned}
\Pr\left((s_1, s_2) = (i, j) \mid (v_1, v_2) = (V_L, V_L)\right) &= r_{0.ij},\\
\Pr\left((s_1, s_2) = (i, j) \mid (v_1, v_2) = (V_H, V_L)\right) &= r_{1.ij},\\
\Pr\left((s_1, s_2) = (i, j) \mid (v_1, v_2) = (V_L, V_H)\right) &= r_{2.ij},\\
\Pr\left((s_1, s_2) = (i, j) \mid (v_1, v_2) = (V_H, V_H)\right) &= r_{ij}.
\end{aligned}$$

We refer to $(r_{0.ij}, r_{1.ij}, r_{2.ij}, r_{ij})_{i \in N, j \in N}$ as a *signal structure*. Of course,

$$\sum_{i=1}^{n}\sum_{j=1}^{n} r_{0.ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} r_{1.ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} r_{2.ij} = \sum_{i=1}^{n}\sum_{j=1}^{n} r_{ij} = 1,$$

$0 \le r_{0.ij} \le 1$, $0 \le r_{1.ij} \le 1$, $0 \le r_{2.ij} \le 1$, and $0 \le r_{ij} \le 1$ for all $(i, j) \in N \times N$. With a slight abuse of terminology, we will refer to a 2-tuple (v_l, s_l) as bidder l’s type. Note that, in general, signal s_l conveys information to bidder l both about bidder m’s $(m = 3 - l)$ valuation, v_m, and about her signal, s_m.

[14] All results easily extend to any $0 \le V_L < V_H$.

[15] In their set up, bidders’ valuations are sometimes allowed to take one of the three possible values. In this case, dealing with three-valued signals already becomes intractable.

Table 1 Distribution of types

	$(V_L, \cdot)$	$(V_H, 1)$	$\cdots$	(V_H, n)
$(V_L, \cdot)$	p^2	$p(1-p)x_{2.1}$	$\cdots$	$p(1-p)x_{2.n}$
$(V_H, 1)$	$p(1-p)x_{1.1}$	$(1-p)^2 r_{11}$	$\cdots$	$(1-p)^2 r_{1n}$
$\vdots$	$\vdots$	$\vdots$		$\vdots$
(V_H, n)	$p(1-p)x_{1.n}$	$(1-p)^2 r_{n1}$	$\cdots$	$(1-p)^2 r_{nn}$

Remark 1 When signals are generated independently of realized valuation profile, $r_{0.ij} = r_{1.ij} = r_{2.ij} = r_{ij}$ for all $(i, j) \in N \times N$. We will refer to such signal structures as correlating devices. The Bayesian Nash equilibria corresponding to a correlating device are known as strategy-correlated equilibria (Cotter 1991; Forges 1986, 1993).

The bidders participate in a first-price auction with zero reserve price,[16] where bidders simultaneously submit bids b depending on the realizations of v and s. The highest bidder gets the object and pays her bid to the seller. In the event of a tie, the bidder with the higher valuation gets the object[17] and the tie-breaking can be arbitrary if bidders' valuations are the same. Next, we present an immediate result about the equilibrium behavior of types with valuation V_L.

Lemma 1 *In any Bayesian Nash equilibrium of the first-price auction, types* (V_L, i) *for all* $i \in N$ *bid* V_L.

The proof is analogous to that in Lemma A.1 of Fang and Morris (2006) and therefore is omitted.[18] Given the result of Lemma 1, in the continuation, we will only need to consider the strategies of high-valuation types. For the same reason, it is irrelevant what signal a low-valuation bidder receives or even if she receives any signal at all. Therefore, the type of a bidder with low valuation can be simply denoted as $(V_L, \cdot)$ for all values of s.

The joint distribution of types is conveniently summarized in Table 1 where

$$x_{1.i} = \sum_{j=1}^{n} r_{1.ij} \text{ and } x_{2.i} = \sum_{j=1}^{n} r_{2.ji}$$

for all $i \in N$. That is, $x_{1.i}$ ($x_{2.i}$, resp.) is the probability that bidder 1 (bidder 2, resp.) with high valuation receives signal i when the opponent has low valuation. For example, the profile of types $((V_H, n), (V_L, \cdot))$ is realized when bidders 1 and 2 have, respectively, high and low valuations, which happens with probability $p(1-p)$, and

16 The assumption of zero reserve price is made purely for simplicity, and all subsequent results extend in the natural way if a binding reserve price is introduced.

17 On the use of this tie-breaking rule, see Maskin and Riley (2000b), Kim and Che (2004), and Fang and Morris (2006). One way to justify it is to assume that in case of a tie, an auxiliary second-price auction is held and the highest bid from the first-price auction serves as the starting price.

18 The proof on page 151 in Maskin and Riley (1985) can also be modified and adopted to prove Lemma 1.

bidder 1 receives signal n, while bidder 2 receives *some* signal, which happens with probability $x_{1.n}$. Similarly, the profile of types $((V_H, 1), (V_H, n))$ is realized when both bidders have high valuations, which happens with probability $(1-p)^2$, and bidder 1 receives signal 1, while bidder 2 receives signal n from the center, which happens with probability r_{1n}. To simplify the notation further, in the continuation, we will refer to types $(V_L, \cdot)$ jointly as type 0, and to type (V_H, i) for each $i \in N$ as type i.

Remark 2 If, instead of maximizing the joint payoff of bidders, we were interested in maximizing seller's revenue, then in the optimum, the center must send either completely uninformative signals or, on the contrary, fully reveal the information about bidders' valuations. In both cases, the ex-ante payoff of a bidder is $p(1-p)$. To see that the seller cannot extract more surplus from the bidders, note that a bidder, according to Lemma 1, can always guarantee an ex-ante payoff of $p(1-p)$ by bidding 0.[19]

3 The optimal two-valued signal structure

We start by considering two-valued symmetric signal structures: $n = 2$, $x_{1.i} = x_{2.i}$ for $i = 1, 2$, and $r_{12} = r_{21}$. Because of the symmetry, we will now drop the subscript that refers to a bidder in $x_{l.i}$ and, instead, write x_i to denote the probability, with which a high-valuation bidder receives signal i when her opponent has low valuation. Following the terminology that we introduced in the previous section, a bidder can be one of the three types: type 0 if she has low valuation; otherwise, she has high valuation and she is either type 1 or type 2 depending on the signal she receives.

We assume that $r_{12} > 0$. If $r_{12} = 0$, then given the equilibrium behavior of type 0 in Lemma 1, the equilibrium strategies of types 1 and 2 can be determined independently. In this case, the derivation of equilibrium follows Maskin and Riley (1985), and one can verify that the ex-ante payoff of bidder is $p(1-p)$, that is, the same as in the absence of collusive communication. Similarly, we rule out the case when

$$\frac{x_1}{x_2} = \frac{r_{11}}{r_{21}} = \frac{r_{12}}{r_{22}}.$$

This case is equivalent to a single uninformative signal, which would again result in the payoff of $p(1-p)$ for each bidder.

In order to find the optimal symmetric signal structure, we first characterize the equilibrium strategies. In particular, we restrict attention to symmetric equilibria.

Proposition 1 *For each two-valued symmetric signal structure, there exists a unique symmetric equilibrium in the first-price auction.*

The full version of proposition, containing the equilibrium strategies, is stated in the "Appendix". Here, we highlight the main properties of the equilibrium.

---

19 The observations made in Remark 2 are not true in general. For example, Landsberger et al. (2001) show that the seller can earn higher revenue by publicly disclosing information about the ranking of valuations among bidders when the valuations are drawn according to a continuous distribution.

Table 2 Cases in Proposition 1

Case 1	Case 3
$\Pr(1\|1) \geq \Pr(1\|2)$ $\frac{\Pr(0\|1)}{\Pr(0\|2)} \geq \frac{\Pr(1\|1)}{\Pr(1\|2)}$	$\Pr(1\|1) < \Pr(1\|2)$ $\Pr(2\|1) \leq \Pr(2\|2)$
Case 2	**Case 4**
$\Pr(1\|1) > \Pr(1\|2)$ $\frac{\Pr(0\|1)}{\Pr(0\|2)} < \frac{\Pr(1\|1)}{\Pr(1\|2)}$	$\Pr(1\|1) < \Pr(1\|2)$ $\Pr(2\|1) > \Pr(2\|2)$

Let $\Pr(j|i)$ denote the conditional probability that a bidder of type i, $i = 1, 2$, assigns to the opponent being of type j, $j = 0, 1, 2$. Since we can always rename the signals, without loss of generality, we assume that $\Pr(0|1) \geq \Pr(0|2)$ holds. Obviously, the equilibrium strategies depend on the signal structure and, consequently, on the conditional probabilities. In particular, when deriving the symmetric equilibrium in Proposition 1, we distinguish four cases that are summarized in Table 2.[20] These cases are mutually exclusive and cover all possibilities.

In the proof to Proposition 1, we show that high-valuation types use mixed strategies in the equilibrium, and the supports of these strategies are intervals. Let $\underline{b}_i$ and $\overline{b}_i$ denote the lower and upper endpoints of the interval for the mixed strategy of type i, $i = 1, 2$. We show that types 1 and 2 bid on adjacent intervals in Cases 1 and 3: $0 = \underline{b}_1 \leq \underline{b}_2 = \overline{b}_1 < \overline{b}_2$. In Case 2, the support of type 1 is a subset of the support of type 2, and both supports have a common lower endpoint: $0 = \underline{b}_1 = \underline{b}_2 < \overline{b}_1 < \overline{b}_2$. In Case 4, the support of type 2 is a subset of the support of type 1, and both supports have a common upper endpoint: $0 = \underline{b}_1 \leq \underline{b}_2 < \overline{b}_1 = \overline{b}_2$.

Before we identify the optimal two-valued signal structure, we first discuss a special case when the signal distribution is constant over the realized valuation profile. That is, we want to know whether there exists a symmetric strategy-correlated equilibrium that improves on the non-cooperative payoffs. When the center uses a correlating device, the probabilities must satisfy the following additional restrictions: $x_1 = r_{11} + r_{12}$ and $x_2 = r_{12} + r_{22}$. They imply that $\Pr(0|1) = \Pr(0|2) = p$. Table 2 then tells us that any strategy-correlated equilibrium falls under Case 2 or 4. One can verify from Proposition 1 that the equilibrium strategies of high-valuation types corresponding to these cases reduce to

$$F_i(b) = \frac{p}{1-p}\frac{b}{1-b}$$

for $b \in [0, 1-p]$, which are the equilibrium strategies of the standard first-price auction (Maskin and Riley 1985).

Corollary 1 *For any two-valued symmetric correlating device, there exists a unique symmetric strategy-correlated equilibrium, which coincides with the unique Bayesian Nash equilibrium of the first-price auction in the absence of any signals.*

[20] If $\Pr(0|2) = 0$ in Case 1, then we set $\Pr(0|1) / \Pr(0|2) = \infty$.

This result motivates us to next study signal structures that depend on bidders' valuations.

Given the equilibrium strategies in Proposition 1, we are in a position to find the symmetric signal structure that maximizes bidders' joint payoff.

Theorem 1 *Among symmetric two-valued signal structures,* $(x_1, x_2, r_{11}, r_{12}, r_{21}, r_{22}) = \left(1, 0, 0, \frac{\sqrt{p}}{1+\sqrt{p}}, \frac{\sqrt{p}}{1+\sqrt{p}}, \frac{1-\sqrt{p}}{1+\sqrt{p}}\right)$ *maximizes bidders' joint payoffs. In the optimum, bidder's ex-ante payoff is* $(1-p)\sqrt{p}$.

The theorem states that the center sends signal 1 for sure ($x_1 = 1, x_2 = 0$) to any high-valuation bidder who faces a low-valuation bidder. If both bidders have high valuations, then the center sometimes sends signal 1 to one and signal 2 to the other bidder ($0 < r_{12} = r_{21} < 1$), and other times he sends signal 2 to both bidders ($0 < r_{22} < 1$). But he never sends signal 1 to both bidders if they both have high valuations ($r_{11} = 0$). For instance, if $p = 0.25$, then the signal profiles $(1, 2)$, $(2, 1)$, and $(2, 2)$ each are sent with a probability of $\frac{1}{3}$ when both bidders have high valuations. The distribution of types induced by the optimal signal structure is shown in Table 3. From the table, it is easy to see that under the optimal signal structure, type 1 is unsure about the opponent's valuation, but she knows that the opponent with high valuation will receive signal 2. On the other hand, type 2 learns that the opponent has high valuation but is unsure about the signal that the opponent observes.

The optimal signal structure falls under Case 3 in Proposition 1, where the support of type 1's strategy has collapsed in a single point.

Corollary 2 *Given the optimal symmetric two-valued signal structure, type 1 bids 0 with probability 1, while type 2 randomizes according to*

$$F_2(b) = \frac{\sqrt{p}}{1-\sqrt{p}} \frac{b}{1-b}$$

on the interval $\left[0, 1-\sqrt{p}\right]$ *in the equilibrium.*

One can easily verify that type 1 is indifferent between bidding 0 and anything in $\left(0, 1-\sqrt{p}\right]$. Intuitively, type 1 knows that if the opponent has high valuation, she will bid relatively aggressively. Therefore, type 1 finds it optimal only to compete against the low valuation opponent. On the other hand, type 2 knows that she faces the high-valuation bidder and therefore indeed needs to bid relatively aggressively. However,

Table 3 Distribution of types given the signal structure in Theorem 1

	$(V_L, \cdot)$	$(V_H, 1)$	$(V_H, 2)$
$(V_L, \cdot)$	p^2	$p(1-p)$	0
$(V_H, 1)$	$p(1-p)$	0	$(1-p)^2 \frac{\sqrt{p}}{1+\sqrt{p}}$
$(V_H, 2)$	0	$(1-p)^2 \frac{\sqrt{p}}{1+\sqrt{p}}$	$(1-p)^2 \frac{1-\sqrt{p}}{1+\sqrt{p}}$

her bid is suppressed by the possibility that the opponent bids 0. This weakens the competition and allows the bidders to earn higher payoffs.

Another way to see where the increased payoffs are coming from is to consider what happens when a bidder bids (just above) 0. In the case when no signals are sent, by bidding 0, the bidder will only win if she faces a low-valuation bidder. Now, additionally, she sometimes wins the auction with bid 0 even if the opponent has high valuation. As an example, suppose $p = 0.25$. In the absence of collusive signals, the bidder earns a payoff of 1 with probability $\frac{3}{16}$ when she has high value and the opponent has low value. Given the optimal signal structure, the bidder still earns a payoff of 1 with probability $\frac{3}{16}$ when she has high value and the opponent has low value, but now she additionally gets a payoff of 1 with probability $\frac{3}{16}$ when both bidders have high value, that is, when she is type 2, while the opponent is type 1 bidder. Note also that both types 1 and 2 expect the same payoff of $\sqrt{p}$ in the optimum. Hence, the effect of the collusive signals is almost the same as increasing the prior of the opponent having low valuation from p to $\sqrt{p}$ in the auction with no signals.

Clearly, the payoff found in Theorem 1 is strictly higher than the payoff of $p(1-p)$ that a bidder would expect in the absence of collusive signals. For example, when $p = 0.25$, the former payoff is double the latter one. If the bidders were able to capture the entire surplus, then the ex-ante payoff of bidder would be $(1-p)\left(p + \frac{1}{2}(1-p)\right)$. Hence, there still exists a scope for improvement upon the optimal two-valued signal payoffs, but we have not been able to prove whether allowing signals to take more values will strictly increase the bidders' payoffs. However, as the next section demonstrates, when the signals are restricted to be public or, equivalently, perfectly correlated, adding more values does not improve on the optimal two-valued public signals.

4 The optimal public signal structure

We now consider a situation when the signals are public. We allow signals to take an arbitrary number of values, $n \geq 2$, and the signal structure to be asymmetric. Consider Table 1. When the signals are public, in each row i, the probabilities r_{ij} are strictly positive for at most one j, and in each column j, the probabilities r_{ij} are strictly positive for at most one i. Since we can reorder and rename rows and columns, we assume that $r_{ij} \geq 0$ if $i = j$ and $r_{ij} = 0$ otherwise. Given this assumption, we say that both bidders observe one common signal, instead of a pair of signals.[21]

Proposition 2 *Given a public signal $i \in N$, if $r_{ii} = 0$, then both bidders submit bids equal to 0. If $x_{1.i} = x_{2.i} = 0$ and $r_{ii} > 0$, then both bidders submit bids equal to 1. If $x_{l.i} \geq x_{m.i}$, $x_{l.i} > 0$, and $r_{ii} > 0$, then the equilibrium of the first-price auction is as follows:*

[21] Said differently, if a public signal i occurs, it is common knowledge that each bidder is of either type (V_L, i) or type (V_H, i). However, given that all types (V_L, i) bid 0, they are suppressed under $(V_L, \cdot)$.

1. Bidder l with $v_l = 1$ *randomizes according to*

$$F_{l.i}(b) = \frac{px_{l.i}}{px_{l.i} + (1-p)\,r_{ii}} \frac{px_{m.i} + (1-p)\,r_{ii}}{(1-p)\,r_{ii}} \frac{1}{1-b} - \frac{px_{m.i}}{(1-p)\,r_{ii}} \tag{1}$$

on the interval $\left[0, \bar{b}_i\right]$ *and puts a mass* $F_{l.i}(0) > 0$ *on bid 0 if* $x_{l.i} > x_{m.i}$*;*

2. Bidder m with $v_m = 1$ *randomizes according to*

$$F_{m.i}(b) = \frac{px_{l.i}}{(1-p)\,r_{ii}} \frac{b}{1-b} \tag{2}$$

on the interval $\left[0, \bar{b}_i\right]$*, where*

$$\bar{b}_i = \frac{(1-p)\,r_{ii}}{px_{l.i} + (1-p)\,r_{ii}}.$$

The equilibrium payoff of a bidder with valuation $v = 1$ *is*

$$\frac{px_{l.i}}{px_{l.i} + (1-p)\,r_{ii}}. \tag{3}$$

Given the equilibrium strategies, we now provide a signal structure that maximizes bidders' joint payoff. In particular, we prove that there exists a two-valued public signal structure that achieves the maximal payoff. In the proof to the following theorem, we show that we can increase the joint payoff if we aggregate all signals for which $x_{1.i} \geq x_{2.i}$ into one single signal and all the remaining signals into the second signal.

Theorem 2 *To achieve the maximal joint payoff with public signals, it is enough that the signal takes two values,* $n = 2$*. The optimal two-valued public signal structure is* $(x_{1.1}, x_{1.2}, x_{2.1}, x_{2.2}, r_{11}, r_{22}) = (1, 0, 0, 1, 0.5, 0.5)$*. In the optimum, each bidder's ex-ante payoff is* $p\,(1-p)\,\frac{2}{1+p}$*.*

The theorem tells that if both bidders have high valuations, then the center sends signal 1 one-half of the time ($r_{11} = 0.5$) and signal 2 the other half of the time ($r_{22} = 0.5$). If only bidder 1 has high value for the object, then the center sends signal 1 for sure ($x_{1.1} = 1$). Similarly, if only bidder 2 has high value, then the center sends signal 2 for sure ($x_{2.2} = 1$). Table 4 summarizes the distribution of types induced by the optimal public signal structure.

Table 4 Distribution of types given the signal structure in Theorem 2

	$(V_L, \cdot)$	$(V_H, 1)$	$(V_H, 2)$
$(V_L, \cdot)$	p^2	0	$p\,(1-p)$
$(V_H, 1)$	$p\,(1-p)$	$\frac{(1-p)^2}{2}$	0
$(V_H, 2)$	0	0	$\frac{(1-p)^2}{2}$

It follows that one of the high-valuation types, call her a weak type, learns that the opponent has high valuation, while the other high-valuation type, call her a strong type, is unsure about the valuation of the opponent. For example, types 1 and 2 are, respectively, the strong and weak types for bidder 1. In this, the structure of optimal public signals is similar to the one found in Theorem 1. On the other hand, a weak type now only bids against a strong type and vice versa, while for the signal structure in Theorem 1, two weak types can bid against each other.

To gain intuition for this result, let us consider what would be the expected payoff of high-valuation bidder in the second-price auction where bidding one's valuation is a dominant strategy. If the bidder was weak, she would expect zero payoff, that is, less than in the absence of the signal. And opposite, if the bidder was strong, she would expect more than in the absence of the signal. In the first-price auction, on the contrary, both weak and strong types expect as much as the strong type would expect in the second-price auction; that is, both types expect more than in the absence of the signal. This suggests that it is optimal to create the highest possible asymmetry between weak and strong types. For example, when $p = 0.25$, then according to Proposition 2, bidder 1 expects a payoff of $\frac{2}{5}$ whether she is type 1 or type 2. In the second-price auction, she would also expect a payoff of $\frac{2}{5}$ but only if she was type 1, which happens with a probability of $\frac{15}{32}$. If she was type 2, her payoff would be zero in the second-price auction.

Note that the optimum can also be implemented with a symmetric public signal structure by exchanging columns corresponding to types 1 and 2 of bidder 2 in Table 4. Therefore, there is no gain in allowing asymmetric public signal structures. Since the optimal public signal structure can be viewed as a special case of signal structures considered in the previous section, the ex-ante payoff of a bidder now is obviously lower than the one found in Theorem 1.

For the signal structure in Theorem 1, the strong type, that is, type 1, is bidding zero in the equilibrium. It is sustained by the fact that the weak type, that is, type 2, is unsure about the signal that the opponent receives. However, with public signals, such uncertainty is absent, and as a result, the strong type must now bid above zero with a positive probability. Therefore, the optimal public signal structure leads to lower payoffs for the bidders compared to the one in Theorem 1. Nevertheless, this signal structure has an attractive feature in that it does not depend on the initial prior. Therefore, if the center does not know p, he might prefer using the optimal public signal structure to facilitate collusion.

5 Independent private signal structures

For the rest of the paper, we assume that the signals are independently and identically distributed: $x_{1.i} = x_{2.i}$ for all $i \in N$, which we write as x_i, and $r_{ij} = y_i y_j$ for all $i, j \in N$. Of course, $\sum_{i \in N} x_i = \sum_{i \in N} y_i = 1$, $0 \leq x_i \leq 1$, and $0 \leq y_i \leq 1$ for all $i \in N$. We assume that there is no $i \in N$ such that $x_i = y_i = 0$. That is, each signal $i \in N$ has ex-ante positive probability to appear; otherwise, we are back to the case with less signals. Without loss of generality, we assume that

$$\frac{x_1}{y_1} > \frac{x_2}{y_2} > \cdots > \frac{x_n}{y_n}.$$

If this relationship is not satisfied, we can always rename the signals. That is, we can always name the signal with the highest ratio as 1, and so on.[22] We prove later that if $\frac{x_i}{y_i} = \frac{x_{i+1}}{y_{i+1}}$ for some i, then signals i and $i+1$ can be considered as one single signal with probabilities $x_{i'} = x_i + x_{i+1}$ and $y_{i'} = y_i + y_{i+1}$, and the corresponding symmetric equilibria are the same in terms of the payoffs. Therefore, we will maintain these assumptions for the rest of the paper. We will now denote a signal structure by $(x, y)_n = (x_j, y_j)_{j \in N}$.

The main result of this section is that once the signals are restricted to be independent, there is no finite valued signal structure that would maximize the bidders' payoffs, meaning, in other words, that it is always possible to increase the payoffs by allowing signals to take more values. First, however, we present the symmetric equilibrium strategies. In the equilibrium, the high-valuation bidder with the lowest signal randomizes on an interval starting at V_L and high-valuation bidders with higher and higher signals randomize on higher and higher adjacent intervals.

Proposition 3 *The unique symmetric equilibrium of the first-price auction with independent private signal structure $(x, y)_n$ is as follows:*

1. *Bidder of type 1 bids 0 if $y_1 = 0$; otherwise, she randomizes over $\left[\bar{b}_0, \bar{b}_1\right]$ according to the cumulative distribution function*

$$F_1(b) = \frac{px_1}{(1-p)\, y_1^2} \frac{b}{1-b},$$

where $\bar{b}_0 \equiv 0$ and

$$\bar{b}_1 = 1 - \frac{px_1}{px_1 + (1-p)\, y_1^2}.$$

2. *Bidder of type i, for $i = 2, \ldots, n$, randomizes over $\left[\bar{b}_{i-1}, \bar{b}_i\right]$ according to the cumulative distribution function*

$$F_i(b) = \frac{px_i + (1-p)\, y_i \sum_{k=1}^{i-1} y_k}{(1-p)\, y_i^2} \frac{b - \bar{b}_{i-1}}{1-b}, \tag{4}$$

where

$$\bar{b}_i = 1 - \frac{px_i + (1-p)\, y_i \sum_{k=1}^{i-1} y_k}{px_i + (1-p)\, y_i \sum_{k=1}^{i} y_k} \left(1 - \bar{b}_{i-1}\right). \tag{5}$$

[22] If $y_i = 0$, then we set $\frac{x_i}{y_i} = \infty$.

Remark 3 If for some $i \in N$, $\frac{x_i}{y_i} = \frac{x_{i+1}}{y_{i+1}}$, then i and $i+1$ have the same expected payoff for any bid in $[\bar{b}_{i-1}, \bar{b}_{i+1}]$. Moreover, it is easy to see that if we replace signals i and $i+1$ with i' having probabilities $x_{i'} = x_i + x_{i+1}$, $y_{i'} = y_i + y_{i+1}$, then in the corresponding equilibrium with $n-1$ signals, it is true that $\bar{b}_{i'} = \bar{b}_{i+1}$ and the strategies of types different from i' do not change. This shows that our original assumption with strict inequalities is indeed without a loss of generality.

The expected payoff of type $i \in N$ is

$$K(i, (x, y)) = \frac{px_i + (1-p)\, y_i \sum_{k=1}^{i-1} y_k}{px_i + (1-p)\, y_i} \left(1 - \bar{b}_{i-1}\right), \tag{6}$$

and each bidder's ex-ante expected payoff is

$$P(x, y) = (1-p) \sum_{i \in N} (px_i + (1-p) y_i) K(i, (x, y)). \tag{7}$$

Example 1 Let $n = 2$ and $x_1 = y_2 = q$ and $x_2 = y_1 = 1 - q$. That is, when signal 1 is equally indicative of value V_L as signal 2 is of value V_H, then the equilibrium described in Proposition 3 is the same as the one in Proposition 1 of Fang and Morris (2006). They show that among these special signal structures, there is an optimal q that minimizes the seller's revenue. However, this signal structure is not optimal for the bidders among all independent private signals. Indeed, the expected payoff of a bidder when there are two signals is

$$\begin{aligned} P(x, y) &= (1-p) px_1 \left(1 + \frac{px_2 + (1-p)\, y_2 y_1}{px_1 + (1-p)\, y_1^2}\right) \\ &= (1-p) px_1 \frac{p + (1-p)\, y_1}{px_1 + (1-p)\, y_1^2}. \end{aligned}$$

The first-order condition with respect to (w.r.t.) x_1 is

$$\frac{p\,(1-p)^2\, y_1^2\, (p + (1-p)\, y_1)}{\left(px_1 + (1-p)\, y_1^2\right)^2},$$

which is strictly positive if $y_1 > 0$.[23] Therefore, it is optimal to set $x_1 = 1$. The first-order condition w.r.t. y_1 is

$$(1-p)\, px_1 \frac{-\left((1-p)\, y_1 + p\right)^2 + p\,(p + (1-p)\, x_1)}{\left(px_1 + (1-p)\, y_1^2\right)^2},$$

which when set equal to 0 and imposing $x_1 = 1$ implies that

[23] If $y_1 = 0$, then $P(x, y) = p\,(1-p)$, that is, the same payoff as in the case when there are no signals at all.

$$y_1 = \frac{\sqrt{p}}{1+\sqrt{p}}.$$

The second derivative w.r.t. y_1 is negative when evaluated at the optimal value of y_1. This indicates that, if signals can take on two different values, $x_1 = y_2 = q$ and $x_2 = y_1 = 1 - q$ does not hold under the optimal independent signal structure.

Bidder's payoff, given the optimal two-valued independent signal structure, is

$$(1-p)\,\frac{\sqrt{p}\left(1+\sqrt{p}\right)}{2}.$$

For example, when $p = 0.25$, the bidder's ex-ante payoff is 0.2813. For comparison, Fang and Morris (2006) show that q that minimizes the seller's revenue is 0.7639, which results in the bidder's payoff of 0.2628.

The following theorem shows what happens with the ex-ante payoff as we increase the number of values that the signals can take.

Theorem 3 *For any n-valued signal structure,* $(x, y)_n$ *there exists an* $(n+1)$*-valued signal structure,* $(x', y')_{n+1}$ *such that the following is true:*

$$P((x, y)_n) < P((x', y')_{n+1}).$$

To prove the theorem,[24] we take an arbitrary $(x, y)_n$, and we show that if $x_n > 0$, then it cannot be optimal. Hence, we assume that $x_n = 0$ and show that we can always introduce an additional signal value that strictly improves the bidder's payoff.

Example 2 To illustrate the approach adopted in the proof of the theorem, consider the optimal two-valued independent signal structure found in Example 1. We construct the three-valued signal structure by reassigning the probabilities as follows:

$$\begin{aligned} x'_1 &= (1-\delta)\,x_1, & y'_1 &= (1-\delta)\,y_1,\\ x'_2 &= \delta x_1, & y'_2 &= \delta y_1 + \epsilon,\\ x'_3 &= 0, & y'_3 &= 1 - y_1 - \epsilon. \end{aligned}$$

Choosing $\delta = 0.5$ and $\epsilon = 0.1$, we find that bidder's ex-ante payoff is 0.2841 when $p = 0.25$. Hence, the bidder's payoff is strictly higher than the one obtained under the optimal two-valued independent signal structure.

Further, the optimal three-valued independent signal structure is[25]

$$\begin{aligned} x_1 &= 0.7295, & y_1 &= 0.1941,\\ x_2 &= 0.2705, & y_2 &= 0.1941,\\ x_3 &= 0, & y_3 &= 0.6118, \end{aligned}$$

which leads to the bidder's payoff of 0.2884.

[24] The proof of this theorem is available in our working paper (Āzacis and Vida 2012).

[25] The optimum was calculated numerically.

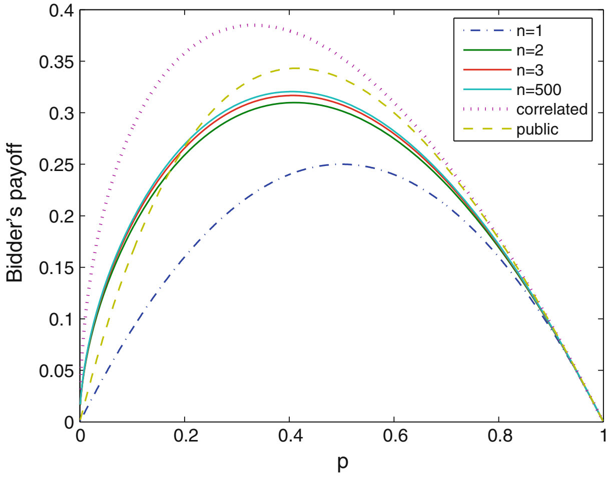

Fig. 1 Bidder's payoff for various signal structures

To find the optimal independent signal structures for $n > 2$ and the corresponding payoffs, we needed to resort to numerical computations. Figure 1 illustrates that although the bidder's payoff strictly increases in n, the gain is small when comparing the optimal payoffs for $n = 2$ and $n = 500$.[26] (Note that $n = 1$ is equivalent to the case when there are no signals at all.) Further, the independent signal structures dominate the optimal public signal structure for low values of p, while the opposite result holds for high values of p. However, it can be verified that on average (that is, assuming that the value of p is drawn from a uniform distribution), the optimal public signal structure outperforms the optimal two-valued independent signal structure. Furthermore, as can be seen from Example 1, the optimal two-valued independent signal structure depends on the knowledge of prior. Numerical results suggest that the same conclusions hold for $n > 2$. These arguments speak in favor of the use of optimal public signal structure to facilitate collusion. Finally, we can see that the optimal two-valued correlated signal structure outperforms all other considered signal structures.

6 Conclusion

We have considered the simplest IPV setup with only two bidders, whose valuations can take either low or high value. After bidders have learnt their own valuations but before submitting their bids in the first-price auction, bidders receive private random signals from a center where the randomness depends on the bidders' realized valuation profile. Given the signals, bidders update their beliefs about their opponent's valuation, form beliefs about the signal received by the opponent, and submit their bids. We have characterized the equilibrium bidding functions and compared bid-

[26] Our numerical calculations suggest that $y_1 = y_2 = \cdots = y_{n-1}$ holds in the optimum. We imposed this constraint in our calculations when n was large.

ders' equilibrium payoffs across different signal structures, and we have found the following.

When the signals can take on one of the two possible values, we have calculated the optimal symmetric (correlated) signal structure. We have shown that the center must use the information about the valuations of bidders; otherwise, the bidders cannot collude at all. That is, there exists no collusive symmetric strategy-correlated equilibrium for two-valued valuation-independent symmetric signal structures. If we restrict our attention to perfectly correlated signals (that is, public signals), the center can attain the optimal payoff with only two-valued public signals. If two signals are chosen independently and identically, the more values the signal can take on the better payoffs bidders can achieve in the symmetric equilibrium. Numerical results suggest that the optimum achieved with two-valued correlated signals is the overall optimum among the signal structures that have been studied. However, the optimal signal structure depends on the bidders' prior beliefs. It is shown that the optimum achieved with public signals is instead independent of the prior. Hence, a center that does not know the bidders' prior might prefer to use public signals to enhance the collusion of bidders.

We have analyzed the case with two possible valuations only. Extending the model from two to more valuations might be challenging. However, we expect some of the features of the two-valuation model to carry over to a more general model. For example, the analysis suggests that the bidders will gain at least sometimes if the center can send valuation-dependent over valuation-independent signals, correlated over independent signals, and private over public signals.

7 Appendix

Proposition 1 *The unique symmetric equilibrium of the first-price auction is as follows:*

Case 1 If $\Pr(1|1) \geq \Pr(1|2) > 0$ *and*

$$\frac{\Pr(0|1)}{\Pr(0|2)} \geq \frac{\Pr(1|1)}{\Pr(1|2)},$$

then

1. type 1 randomizes on $\left[0, \bar{b}_1\right]$ *according to*

$$F_1(b) = \frac{\Pr(0|1)}{\Pr(1|1)} \frac{b}{1-b} \tag{8}$$

where

$$\bar{b}_1 = \frac{\Pr(1|1)}{\Pr(0|1) + \Pr(1|1)}; \tag{9}$$

2. *type 2 randomizes on* $\left[\overline{b}_1, \overline{b}_2\right]$ *according to*

$$F_2(b) = \frac{\Pr(0|2) + \Pr(1|2)}{\Pr(2|2)} \frac{b - \overline{b}_1}{1 - b} \tag{10}$$

where

$$\bar{b}_2 = 1 - (\Pr(0|2) + \Pr(1|2))\left(1 - \bar{b}_1\right). \tag{11}$$

Case 2 If $\Pr(1|1) > \Pr(1|2) > 0$ *and*

$$\frac{\Pr(1|1)}{\Pr(1|2)} > \frac{\Pr(0|1)}{\Pr(0|2)},$$

then

1. *type 1 randomizes on* $\left[0, \overline{b}_1\right]$ *according to*

$$F_1(b) = \frac{\Pr(2|2)\Pr(0|1) - \Pr(2|1)\Pr(0|2)}{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)} \frac{b}{1 - b}$$

where

$$\overline{b}_1 = \frac{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)}{\Pr(2|2) - \Pr(2|1)};$$

2. *type 2 randomizes on* $\left[0, \overline{b}_1\right]$ *according to*

$$F_2(b) = \frac{\Pr(1|1)\Pr(0|2) - \Pr(1|2)\Pr(0|1)}{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)} \frac{b}{1 - b}$$

and on $\left[\overline{b}_1, \overline{b}_2\right]$ *according to*

$$F_2(b) = \frac{\Pr(0|2)}{\Pr(2|2)} \frac{b}{1 - b} - \frac{\Pr(1|2)}{\Pr(2|2)}$$

where $\overline{b}_2 = 1 - \Pr(0|2)$.

Case 3 If $\Pr(1|1) < \Pr(1|2)$ *and* $0 < \Pr(2|1) \leq \Pr(2|2)$, *then the types bid as in Case* 1.

Case 4 If $\Pr(1|1) < \Pr(1|2)$ *and* $\Pr(2|1) > \Pr(2|2)$, *then*

1. *type 1 randomizes on* $\left[0, \underline{b}_2\right]$ *according to* (8), *where*

$$\underline{b}_2 = \frac{\Pr(1|1)(\Pr(0|1) - \Pr(0|2))}{\Pr(1|2)\Pr(0|1) - \Pr(1|1)\Pr(0|2)},$$

and on $\left[\underline{b}_2, \overline{b}_2\right]$ *according to*

$$F_1(b) = \frac{\Pr(0|1)(\Pr(2|2) - \Pr(2|1))}{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)}\frac{1}{1-b} + \frac{\Pr(2|1)\Pr(0|2) - \Pr(2|2)\Pr(0|1)}{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)}$$

where $\overline{b}_2 = 1 - \Pr(0|1)$*;*

2. *type 2 randomizes on* $\left[\underline{b}_2, \overline{b}_2\right]$ *according to*

$$F_2(b) = \frac{\Pr(0|1)(\Pr(1|1) - \Pr(1|2))}{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)}\frac{1}{1-b} + \frac{\Pr(1|2)\Pr(0|1) - \Pr(1|1)\Pr(0|2)}{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)}.$$

Proof of Proposition 1 Without loss of generality, let $\Pr(0|1) \geq \Pr(0|2)$. We also assume that $\Pr(2|1) > 0$, $\Pr(1|2) > 0$, and $\Pr(k|1) \neq \Pr(k|2)$ for some $k \in \{0, 1, 2\}$ hold. As discussed in the main text, if these assumptions are not satisfied, the expected payoff of bidder is $p(1-p)$.

It is easy to argue that the supremum of all submitted bids does not exceed 1 in any equilibrium. Also, by bidding 1, a type i bidder, $i = 1, 2$, expects a payoff of 0. To sustain it as a part of equilibrium, she must assign zero probability to the opponent bidding anything less than 1. However, given that $x_1 + x_2 = 1$ and $r_{12} > 0$ holds, there is a type of opponent who will bid for sure strictly less than 1.[27] It follows that the supremum of all submitted bids is strictly less than 1, and each type i bidder, $i = 1, 2$, expects a strictly positive payoff and wins with a strictly positive probability in any equilibrium.

Suppose now that bidder 2 of a high-valuation type bids $0 < \tilde{b} < 1$ with a strictly positive probability in an equilibrium. Then, bidder 1 of either high-valuation type, instead of bidding in the interval $\left[\tilde{b} - \delta, \tilde{b}\right]$ for some $\delta > 0$, is better off by bidding $\tilde{b} + \epsilon$ where $\epsilon > 0$ is sufficiently small. But then bidder 2 is better off to bid $\tilde{b} - \delta$ instead of $\tilde{b}$, which contradicts our assumption that she bids $0 < \tilde{b} < 1$ with a strictly positive probability. Thus, only the bid of 0 can possibly occur with a strictly positive probability. Further, the high-valuation bidders cannot tie with a positive probability at 0 in the equilibrium, as either bidder will instead prefer to bid just above 0. In particular, it implies that if type i bids 0 with a positive probability, then $r_{ii} = 0$.

Let $F_i(b)$ be the distribution function according to which type i, $i = 1, 2$, randomizes in an equilibrium. Consider the union of supports of $F_1(b)$ and $F_2(b)$. We claim that this union of supports is a connected set. Suppose to the contrary that bidder 2 with high valuation bids $\tilde{b}$ but does not bid in the interval $\left[\tilde{b} - \delta, \tilde{b}\right)$. But then bidder 1 with high valuation prefers bidding $\tilde{b} - \delta$ instead of $\tilde{b}$. In the same way, we can argue that the lower limit of the union of supports is 0.

[27] If, for example, $x_2 = r_{12} = 0$, then type 2 would bid 1 with probability 1 in the equilibrium.

Since there are no ties between high-valuation bidders in equilibrium, the expected payoff of type i bidder from submitting a bid b is given by

$$\left\{\Pr(0|i) + \Pr(i|i)\, F_i(b) + \Pr(j|i)\, F_j(b)\right\}(1-b). \tag{12}$$

Let π_i denote the expected payoff of type i, $i = 1, 2$. Equation (12) implies that $\pi_i \geq \Pr(0|i)$. We have already argued that there is a type i such that $\underline{b}_i = 0$. Therefore, either $\pi_1 = \Pr(0|1)$ or $\pi_2 = \Pr(0|2)$ or both must hold. If $\pi_i = \Pr(0|i)$ for $i = 1, 2$, then the ex-ante payoffs are the same as in the absence of collusive communication. If $\overline{b}_1 = \overline{b}_2$, then $\pi_1 = \pi_2 = \max\{\Pr(0|1), \Pr(0|2)\} = \Pr(0|1)$. If $\overline{b}_i > \overline{b}_j$, then $\pi_j \geq \pi_i = 1 - \overline{b}_i$.

Now, we derive what are the restrictions on the conditional probabilities for one or another equilibrium configuration to exist, and next, using the identified restrictions, we determine what is the equilibrium configuration for each signal structure.

Suppose there is an equilibrium in which $0 = \underline{b}_i < \underline{b}_j$ holds. Then, $\Pr(i|i) > 0$, and the expected payoff of type i from bidding on the interval $\left[0, \underline{b}_j\right]$ is

$$\{\Pr(0|i) + \Pr(i|i)\, F_i(b)\}(1-b) = \Pr(0|i),$$

which implies that

$$F_i(b) = \frac{\Pr(0|i)}{\Pr(i|i)} \frac{b}{1-b} \tag{13}$$

for $0 \leq b \leq \underline{b}_j$. Consider now type j, who deviates and bids below $\underline{b}_j$. The expected payoff is

$$\begin{aligned} &\left\{\Pr(0|j) + \Pr(i|j) \frac{\Pr(0|i)}{\Pr(i|i)} \frac{b}{1-b}\right\}(1-b) \\ &\quad = \Pr(0|j) + \left\{\Pr(i|j) \frac{\Pr(0|i)}{\Pr(i|i)} - \Pr(0|j)\right\} b. \end{aligned}$$

It must be that the term in curly brackets is positive, or otherwise, type j will find it profitable to deviate. Hence,

$$\frac{\Pr(0|i)}{\Pr(0|j)} \geq \frac{\Pr(i|i)}{\Pr(i|j)} \tag{14}$$

must hold.

Suppose there is an equilibrium in which $\overline{b}_i > \overline{b}_j$ holds. Then, $\Pr(i|i) > 0$, and the expected payoff of type i from bidding on the interval $\left[\overline{b}_j, \overline{b}_i\right]$ is

$$\{\Pr(0|i) + \Pr(j|i) + \Pr(i|i)\, F_i(b)\}(1-b) = 1 - \overline{b}_i,$$

which implies that

$$F_i(b) = \frac{1}{\Pr(i|i)} \left[\frac{1-\overline{b}_i}{1-b} - (\Pr(0|i) + \Pr(j|i)) \right] \tag{15}$$

for $b \in [\overline{b}_j, \overline{b}_i]$. Consider now type j, who deviates and bids above $\overline{b}_j$. The expected payoff is

$$\left\{ \Pr(0|j) + \Pr(j|j) + \frac{\Pr(i|j)}{\Pr(i|i)} \left[\frac{1-\overline{b}_i}{1-b} - (\Pr(0|i) + \Pr(j|i)) \right] \right\} (1-b)$$
$$= \frac{\Pr(i|j)}{\Pr(i|i)} \left(1-\overline{b}_i\right) + \left\{ \Pr(0|j) + \Pr(j|j) - \frac{\Pr(i|j)}{\Pr(i|i)} (\Pr(0|i) + \Pr(j|i)) \right\} (1-b).$$

It must be that the term in curly brackets is positive, or otherwise, type j will find it profitable to deviate. Hence,

$$\frac{\Pr(0|j) + \Pr(j|j)}{\Pr(i|j)} \geq \frac{\Pr(0|i) + \Pr(j|i)}{\Pr(i|i)}$$

or

$$\Pr(i|i) \geq \Pr(i|j) \tag{16}$$

must hold.

Suppose that $0 = \underline{b}_i \leq \underline{b}_j < \overline{b}_i \leq \overline{b}_j$, that is, there exists an interval, in which both types bid in an equilibrium. For the bids in this interval, we can set the expression in (12) equal to π_i and multiply both sides with $\Pr(j|j)$. Similarly, we multiply the analogous expression for type j with $\Pr(j|i)$:

$$\Pr(j|j)\,\pi_i = \Pr(j|j) \left\{ \Pr(0|i) + \Pr(i|i)\, F_i(b) + \Pr(j|i)\, F_j(b) \right\} (1-b),$$
$$\Pr(j|i)\,\pi_j = \Pr(j|i) \left\{ \Pr(0|j) + \Pr(i|j)\, F_i(b) + \Pr(j|j)\, F_j(b) \right\} (1-b).$$

Subtracting the second line from the first and rearranging, we obtain the expression for $F_i(b)$ in the region where the supports intersect:

$$F_i(b) = \frac{\Pr(j|j)\,\pi_i - \Pr(j|i)\,\pi_j}{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)} \frac{1}{1-b} + \frac{\Pr(j|i)\Pr(0|j) - \Pr(j|j)\Pr(0|i)}{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)}. \tag{17}$$

The conditional probabilities and payoffs must be such that $F_i(b)$ is a strictly increasing function in this region. Also, note that if $\Pr(2|2)\Pr(1|1) = \Pr(2|1)\Pr(1|2)$, then the supports of both types cannot overlap.

We are now in the position to determine the equilibria for all possible values of conditional probabilities. To simplify the analysis, we initially find a symmetric equilibrium, in which the support of bids for each type is connected. Afterward, we will argue that it is the unique symmetric equilibrium by showing that there are no symmetric equilibria, in which the support for either type is disconnected.

Case 1 $\Pr(1|1) \geq \Pr(1|2) > 0$ and

$$\frac{\Pr(0|1)}{\Pr(0|2)} \geq \frac{\Pr(1|1)}{\Pr(1|2)} \tag{18}$$

imply that $0 < \Pr(2|1) < \Pr(2|2)$ and

$$\frac{\Pr(0|2)}{\Pr(0|1)} < \frac{\Pr(2|2)}{\Pr(2|1)} \tag{19}$$

also hold. According to (14), conditions (18) and (19), in turn, imply that $0 = \underline{b}_1 \leq \underline{b}_2$ must hold. We argue by contradiction that $\underline{b}_2 < \bar{b}_1$ cannot happen in the equilibrium. If $\pi_1 \geq \Pr(0|1)$ and $\pi_2 = \Pr(0|2)$, then $F_2(b)$ is a non-increasing function in (17) for $(i, j) = (2, 1)$. Therefore, if $\underline{b}_2 < \bar{b}_1$, then $\pi_1 = \Pr(0|1)$ and $\pi_2 > \Pr(0|2)$, which also implies that $F_2\left(\underline{b}_2\right) = 0$. However, evaluating (17) for $(i, j) = (2, 1)$ at $\underline{b}_2$, we find that

$$1 - \underline{b}_2 = \frac{\Pr(1|1)\,\pi_2 - \Pr(1|2)\Pr(0|1)}{\Pr(1|1)\Pr(0|2) - \Pr(1|2)\Pr(0|1)} > 1,$$

or $\underline{b}_2 < 0$. Thus, it follows that $\bar{b}_1 = \underline{b}_2$ must hold; that is, the types bid on adjacent intervals. Thus, type 1 bids on $\left[0, \bar{b}_1\right]$ according to (13) or (8), where $\bar{b}_1$ is defined by $F_1\left(\bar{b}_1\right) = 1$ and is given in (9). Type 2 bids on $\left[\bar{b}_1, \bar{b}_2\right]$ according to (15). Since type 2 must be indifferent between bidding $\bar{b}_1$ and $\bar{b}_2$, it follows that $\bar{b}_2$ is given by (11). This allows to rewrite (15) into the form given in (10).

Case 2 $\Pr(1|1) > \Pr(1|2) > 0$ and

$$\frac{\Pr(1|1)}{\Pr(1|2)} > \frac{\Pr(0|1)}{\Pr(0|2)} \tag{20}$$

imply that $0 < \Pr(2|1) < \Pr(2|2)$ and

$$\frac{\Pr(0|2)}{\Pr(0|1)} < \frac{\Pr(2|2)}{\Pr(2|1)} \tag{21}$$

also hold. According to (14), conditions (20) and (21) imply that it must be the case that $\underline{b}_1 = \underline{b}_2 = 0$. If $0 = \underline{b}_i < \underline{b}_j$, then type j would have incentives to bid below $\underline{b}_j$. $\Pr(1|1) > 0$ and $\Pr(2|2) > 0$ imply that there cannot be a mass point at 0, and the payoffs are $\pi_1 = \Pr(0|1)$ and $\pi_2 = \Pr(0|2)$, which results in the same ex-ante payoffs as in the case without collusive communication. Further, given the assumed conditions on probabilities, we can verify that the distribution function in (17) is well-defined for $i = 1, 2$; that is, it is positively sloped. Also, according to (17), the values of b at which $F_1(b) = 1$ and $F_2(b) = 1$ hold, respectively, are

$$b = \frac{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)}{\Pr(2|2) - \Pr(2|1)},$$
$$b = \frac{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)}{\Pr(1|1) - \Pr(1|2)}.$$

Since the former value of b is smaller than the latter, it means that $F_1(b)$ reaches 1 earlier. Therefore,

$$\bar{b}_1 = \frac{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)}{\Pr(2|2) - \Pr(2|1)} > 0$$

and $\bar{b}_2 = 1 - \pi_2 = 1 - \Pr(0|2) \geq \bar{b}_1$. Further, one can verify that distribution functions (15) and (17) for $(i, j) = (2, 1)$ evaluated at $\bar{b}_1$ give the same answer:

$$F_2\left(\bar{b}_1\right) = \frac{\Pr(0|2)\Pr(1|1) - \Pr(0|1)\Pr(1|2)}{\Pr(2|2)\Pr(0|1) - \Pr(2|1)\Pr(0|2)} > 0.$$

To summarize, in the equilibrium, the types 1 and 2 bid according to (17), where $\pi_1 = \Pr(0|1)$ and $\pi_2 = \Pr(0|2)$, on the interval $\left[0, \bar{b}_1\right]$, and type 2 also bids on $\left[\bar{b}_1, \bar{b}_2\right]$ according to (15).

Case 3 According to (16), conditions $\Pr(1|1) < \Pr(1|2)$ and $\Pr(2|1) \leq \Pr(2|2)$ imply that $\bar{b}_1 \leq \bar{b}_2$ holds. But then $\pi_2 = 1 - \bar{b}_2$, and it must be the case that $\pi_1 \geq \pi_2$ or otherwise type 1 would deviate and bid $\bar{b}_2$. We argue that $\underline{b}_2 < \bar{b}_1$ cannot happen in the equilibrium. Conditions $\Pr(1|1) < \Pr(1|2)$ and $\Pr(2|1) \leq \Pr(2|2)$ together with $\pi_1 \geq \pi_2$ imply that $F_i(b)$ in (17) is a decreasing function for one of the types, depending on the sign of $\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)$. Therefore, $\bar{b}_1 = \underline{b}_2$ must hold, and the types bid as in Case 1.

Case 4 According to (16), conditions $\Pr(1|1) < \Pr(1|2)$ and $\Pr(2|1) > \Pr(2|2)$ imply that there cannot be an equilibrium in which $\bar{b}_i > \bar{b}_j$, as type j will want to deviate and bid above $\bar{b}_j$. Hence, if there is an equilibrium, $\bar{b}_1 = \bar{b}_2$ must hold. This in turn implies that $\pi_1 = \pi_2$. As we know that $\pi_i \geq \Pr(0|i)$ for $i = 1, 2$ and there exists a type j for whom $\pi_j = \Pr(0|j)$, it must be the case that $\pi_1 = \pi_2 = \Pr(0|1)$. This in turn implies that $F_2\left(\underline{b}_2\right) = 0$, and $0 = \underline{b}_1 \leq \underline{b}_2$. Also, since $\pi_i = 1 - \bar{b}_i$ for $i = 1, 2$, then $\bar{b}_1 = \bar{b}_2 = 1 - \Pr(0|1)$.

Since $F_2\left(\underline{b}_2\right) = 0$, from (17) for $(i, j) = (2, 1)$, we can solve out for $\underline{b}_2$:

$$\underline{b}_2 = \frac{\Pr(1|1)\left(\Pr(0|1) - \Pr(0|2)\right)}{\Pr(1|2)\Pr(0|1) - \Pr(1|1)\Pr(0|2)} \geq 0.$$

Further, from (17) for $(i, j) = (1, 2)$, we find that

$$F_1\left(\underline{b}_2\right) = \frac{\Pr(0|1) - \Pr(0|2)}{\Pr(1|2) - \Pr(1|1)} \geq 0. \tag{22}$$

Also note that (13) for $i = 1$ gives the same expression for $F_1(\underline{b}_2)$. We can also verify that $F_1(\underline{b}_2) < 1$. It holds if $\Pr(0|1) + \Pr(1|1) < \Pr(0|2) + \Pr(1|2)$, which is indeed true as it is equivalent to $\Pr(2|1) > \Pr(2|2)$. Finally, one can also verify that $\underline{b}_2 < \overline{b}_1 = \overline{b}_2$. Hence, we have characterized the equilibrium, in which type 1 bids according to (13) in $[0, \underline{b}_2]$, and both types bid according to (17), where $\pi_1 = \pi_2 = \Pr(0|1)$, on the interval $[\underline{b}_2, \overline{b}_2]$.

Finally, we prove that in any symmetric equilibrium, the support of equilibrium bids for each type must be connected. Suppose to the contrary that there exists an interval $(\underline{b}, \overline{b})$ such that only type i bids in this interval, while type j (and possibly type i) bids on two disconnected intervals, and $\underline{b}$ is the upper limit of the first of these intervals, while $\overline{b}$ is the lower limit of the second of these intervals. Note that $F_j(\underline{b}) = F_j(\overline{b})$ and $\Pr(i|i) > 0$ hold. The expected payoff of type i from bidding in the interval $(\underline{b}, \overline{b})$ is

$$\pi_i = \left\{\Pr(0|i) + \Pr(j|i) F_j(\underline{b}) + \Pr(i|i) F_i(b)\right\}(1 - b),$$

which implies that

$$F_i(b) = \frac{1}{\Pr(i|i)}\left[\frac{\pi_i}{1 - b} - \left(\Pr(0|i) + \Pr(j|i) F_j(\underline{b})\right)\right].$$

Consider now type j, who deviates and bids in $(\underline{b}, \overline{b})$. The expected payoff is

$$\begin{aligned}&\left\{\Pr(0|j)+\Pr(j|j) F_j(\underline{b})+\frac{\Pr(i|j)}{\Pr(i|i)}\left[\frac{\pi_i}{1-b}-\left(\Pr(0|i)+\Pr(j|i) F_j(\underline{b})\right)\right]\right\}(1-b)\\ &=\frac{\Pr(i|j)}{\Pr(i|i)}\pi_i+\left\{\Pr(0|j)+\Pr(j|j) F_j(\underline{b})-\frac{\Pr(i|j)}{\Pr(i|i)}(\Pr(0|i)\right.\\ &\quad\left.+\Pr(j|i) F_j(\underline{b}))\right\}(1-b).\end{aligned}$$

The above expression is linear in b. Further, note that in equilibrium, type j is indifferent between bidding $\underline{b}$ and $\overline{b}$. Therefore, he must be indifferent among all bids in the interval $(\underline{b}, \overline{b})$, which implies that the expression in the curly brackets is zero. Then,[28]

$$F_j(\underline{b}) = F_j(\overline{b}) = \frac{\Pr(0|i)\Pr(i|j) - \Pr(0|j)\Pr(i|i)}{\Pr(2|2)\Pr(1|1) - \Pr(2|1)\Pr(1|2)}, \tag{23}$$

and

$$\pi_j = \frac{\Pr(i|j)}{\Pr(i|i)}\pi_i. \tag{24}$$

It immediately follows from (23) that there is at most one discontinuity in the support for each type.

[28] Alternatively, the expression in the curly brackets is zero if $\Pr(k|1) = \Pr(k|2)$ for all k, but this case has been ruled out.

We again consider all four cases.

Case 1 According to $\Pr(1|1) \geq \Pr(1|2)$, $\Pr(2|1) < \Pr(2|2)$, and (19), it is true that $F_1(\underline{b}) < 0$ for $(i, j) = (2, 1)$ in (23). Therefore, the support of bids for type 1 is connected. Further, from the previous analysis of Case 1, we already know that $\underline{b}_1 = 0$. It means that type $i = 1$ must bid not only in the interval $(\underline{b}, \overline{b})$, but also in $[0, \underline{b}]$. But then there is an interval, in which both types bid, which has already been ruled out by the previous analysis. Hence, it must be that the support of bids for type $j = 2$ is also connected.

Case 2 According to $\Pr(1|1) > \Pr(1|2)$, $\Pr(2|1) < \Pr(2|2)$, (20), and (21), we obtain that $F_1(\underline{b}) < 0$ and $F_2(\underline{b}) < 0$ in (23), which is a contradiction. Therefore, the support of bids for each type must be connected.

Case 3 We already know that the supports of both types cannot overlap and $\overline{b}_1 < \overline{b}_2$ holds. Therefore, if the support of bids for type 1 is disconnected, then so is the support of bids for type 2. Thus, it is sufficient to argue that the support of bids for type 2 must be connected. We already know that $\pi_1 \geq \pi_2$. However, Eq. (24) for $(i, j) = (1, 2)$ together with $\Pr(1|1) < \Pr(1|2)$ implies that $\pi_1 < \pi_2$. Hence, we have obtained a contradiction.

Case 4 We know that in any equilibrium $\overline{b}_1 = \overline{b}_2$ and $\pi_1 = \pi_2$ must hold. Then, (24) implies that $\Pr(i|j) = \Pr(i|i)$, which contradicts the assumptions about conditional probabilities that $\Pr(1|1) < \Pr(1|2)$ and $\Pr(2|1) > \Pr(2|2)$.

This completes the proof that the support of bids must always be connected for each type in a symmetric equilibrium. This also implies that the equilibrium that we have found before is the unique symmetric equilibrium for each distribution of signals.

Proof of Theorem 1 We structure the proof along the cases identified in Proposition 1.

Cases 1 *and* 3 These cases result in the same equilibrium structure whereby the types bid on adjacent intervals. Combining them, the restrictions on probabilities such that the types bid on adjacent intervals are

$$\Pr(0|1) \geq \Pr(0|2),$$
$$\Pr(2|2) \geq \Pr(2|1),$$
$$\frac{\Pr(0|1)}{\Pr(1|1)} \geq \frac{\Pr(0|2)}{\Pr(1|2)},$$

or equivalently,

$$\frac{px_1}{px_1 + (1-p)(r_{11} + r_{12})} \geq \frac{px_2}{px_2 + (1-p)(r_{12} + r_{22})},$$
$$\frac{(1-p)r_{22}}{px_2 + (1-p)(r_{12} + r_{22})} \geq \frac{(1-p)r_{12}}{px_1 + (1-p)(r_{11} + r_{12})},$$
$$\frac{x_1}{r_{11}} \geq \frac{x_2}{r_{12}}.$$

The payoff of high-valuation bidder before she receives a signal is

$$\begin{aligned}
&\Pr(1)\,\pi_1 + \Pr(2)\,\pi_2 \\
&\quad= \Pr(1)\Pr(0|1) + \Pr(2)\left(1-\bar{b}_2\right) \\
&\quad= \Pr(1)\Pr(0|1) + \Pr(2)\left(\Pr(0|2)+\Pr(1|2)\right)\frac{\Pr(0|1)}{\Pr(0|1)+\Pr(1|1)} \\
&\quad= px_1\left(1+\frac{px_2+(1-p)\,r_{12}}{px_1+(1-p)\,r_{11}}\right) \\
&\quad= px_1\frac{p+(1-p)\,(r_{11}+r_{12})}{px_1+(1-p)\,r_{11}},
\end{aligned} \tag{25}$$

where we have used (9) and (11), and $\Pr(i)$ for $i = 1, 2$ denotes the probability that the bidder will be of type i.

Hence, the problem that we are solving is[29]

$$\max_{x_1,x_2,r_{11},r_{12},r_{22}} px_1\frac{p+(1-p)\,(r_{11}+r_{12})}{px_1+(1-p)\,r_{11}} \tag{P1}$$

subject to

$$\begin{aligned}
x_1+x_2 &= 1, \\
r_{11}+2r_{12}+r_{22} &= 1, \\
\frac{x_1}{r_{11}} &\geq \frac{x_2}{r_{12}}, \\
\frac{px_1}{px_1+(1-p)\,(r_{11}+r_{12})} &\geq \frac{px_2}{px_2+(1-p)\,(r_{12}+r_{22})}, \\
\frac{(1-p)\,r_{22}}{px_2+(1-p)\,(r_{12}+r_{22})} &\geq \frac{(1-p)\,r_{12}}{px_1+(1-p)\,(r_{11}+r_{12})},
\end{aligned}$$

and all probabilities $(x_1, x_2, r_{11}, r_{12}, r_{22})$ must be non-negative. The non-negativity constraints together with the above equality constraints ensure that each of the probabilities is also less than 1. The inspection of this program tells that the objective function is increasing in x_1. Therefore, setting $x_1 = 1$ and $x_2 = 0$ is optimal, since it does not violate any of the constraints. Using these results, we simplify our problem to

$$\max_{r_{11},r_{12},r_{22}} \frac{r_{12}}{p+(1-p)\,r_{11}}$$

subject to

$$\begin{aligned}
r_{11}+2r_{12}+r_{22} &= 1, \\
r_{22}\,(p+(1-p)\,r_{11}) &\geq (1-p)\,r_{12}^2,
\end{aligned}$$

[29] Since we restrict attention to symmetric signal structures and symmetric equilibria, it is enough to maximize the payoff of single bidder.

and (r_{11}, r_{12}, r_{22}) must be non-negative. Note that we have taken a monotone transformation of the objective function.

Suppose the above inequality does not bind. Then, it is always possible to increase the payoff by raising r_{12} by a small amount, and correspondingly decreasing r_{22}. (If $r_{22} = 0$, then $r_{12} = 0$ must also hold. But then the objective also takes zero value, which is clearly not a maximum.) Therefore, $r_{22}(p + (1 - p) r_{11}) = (1 - p) r_{12}^2$ holds. Using both equalities, we can solve for r_{11} and r_{12} as functions of r_{22}

$$r_{11} = 1 - 2\frac{\sqrt{r_{22}}}{\sqrt{1 - p}} + r_{22}, \tag{26}$$

$$r_{12} = \frac{\sqrt{r_{22}}}{\sqrt{1 - p}} - r_{22}, \tag{27}$$

and write the objective function as

$$\max_{r_{22}} \frac{\sqrt{r_{22}}}{\frac{1}{\sqrt{1-p}} - \sqrt{r_{22}}},$$

which is always increasing in r_{22}.

To now, we have ignored the non-negativity constraints that $r_{11} \geq 0$ and $r_{12} > 0$ must also be satisfied. One can verify from (27) that $r_{12} > 0$ is satisfied for all $0 < r_{22} \leq 1$, while from (26), r_{11} is always decreasing in r_{22}. Therefore, r_{22} can be raised up to the point where $r_{11} = 0$, implying that

$$r_{12} = \frac{\sqrt{p}}{1 + \sqrt{p}},$$

$$r_{22} = \frac{1 - \sqrt{p}}{1 + \sqrt{p}}.$$

Evaluating (25) at the optimum gives that the payoff of high-valuation bidder before she receives a signal is $\sqrt{p}$. Therefore, bidder's ex-ante payoff is $(1 - p)\sqrt{p}$ in the optimum.

Case 2 The payoffs of types 1 and 2 are, respectively, $\pi_1 = \Pr(0|1)$ and $\pi_2 = \Pr(0|2)$; therefore, the ex-ante payoff of bidder is the same as in the case without collusive communication, $p(1 - p)$.

Case 4 The payoff of high-valuation bidder in this equilibrium is $\Pr(0|1)$, irrespective of her type. Hence, we are solving the following problem:

$$\max \Pr(0|1)$$

subject to $\Pr(0|1) \geq \Pr(0|2)$, $\Pr(1|1) < \Pr(1|2)$, and $\Pr(2|1) > \Pr(2|2)$. Constraint $\Pr(1|1) < \Pr(1|2)$ can be ignored as it is implied by the other two inequalities. Thus,

$$\max_{x_1,x_2,r_{11},r_{12},r_{22}} \frac{px_1}{px_1+(1-p)(r_{11}+r_{12})} \tag{P2}$$

subject to

$$\begin{aligned} x_1+x_2&=1,\\ r_{11}+2r_{12}+r_{22}&=1,\\ \frac{px_1}{px_1+(1-p)(r_{11}+r_{12})}&\geq\frac{px_2}{px_2+(1-p)(r_{12}+r_{22})},\\ \frac{(1-p)r_{12}}{px_1+(1-p)(r_{11}+r_{12})}&>\frac{(1-p)r_{22}}{px_2+(1-p)(r_{12}+r_{22})}, \end{aligned}$$

and all probabilities $(x_1, x_2, r_{11}, r_{12}, r_{22})$ must be non-negative. The objective is increasing in x_1 and decreasing in r_{11} and r_{12}. Consider decreasing r_{11} by 2δ, while increasing r_{12} by δ. Then, none of the constraints is violated and the objective has increased. Therefore, it is optimal to set $r_{11}=0$. We can rewrite the inequality constraints as

$$\frac{px_2}{px_2+(1-p)(r_{12}+r_{22})}\leq\frac{px_1}{px_1+(1-p)r_{12}}<\frac{px_2+(1-p)r_{12}}{px_2+(1-p)(r_{12}+r_{22})}.$$

We want

$$\frac{px_1}{px_1+(1-p)r_{12}}$$

to be as high as possible but strictly less than the right-most expression in the above constraint. It follows that this program does not have a maximum. If we sat the second inequality as equality in the above constraint, we would obtain that $\Pr(0|1)=1-\Pr(2|2)$. This, together with $\Pr(1|1)=0$, implies that $F_1\left(\underline{b}_2\right)=1$ in (22); that is, the types bid on adjacent intervals, which contradicts the equilibrium structure of Case 4.

Although program (P2) does not have a maximum, we still need to verify that its supremum does not exceed the maximum that we have found for program (P1). To find the supremum, we can rewrite (P2) in the following form:

$$\max_{x_1,x_2,r_{12},r_{22}} p+(1-p)r_{12} \tag{P3}$$

subject to

$$\begin{aligned} x_1+x_2&=1,\\ 2r_{12}+r_{22}&=1,\\ \frac{px_1}{px_1+(1-p)r_{12}}&\geq\frac{px_2}{px_2+(1-p)(r_{12}+r_{22})},\\ \frac{px_1}{px_1+(1-p)r_{12}}&=\frac{px_2+(1-p)r_{12}}{px_2+(1-p)(r_{12}+r_{22})}, \end{aligned}$$

and all probabilities $(x_1, x_2, r_{12}, r_{22})$ must be non-negative, where the objective of (P3) is obtained by combining the three equality constraints to express x_1 as a function of r_{12}. At the same time, we know that the solution to program (P1) satisfies $r_{11} = 0$ and the third inequality holds as equality. If we impose these constraints on (P1) from the outset, we obtain program (P3). (The first inequality in (P1) is automatically satisfied when $r_{11} = 0$ and therefore can be ignored.) Hence, we conclude that (P1) and (P3) have the same solution. This completes the proof that the optimal signal structure is given as the solution to program (P1).

Proof of Proposition 2 Given a public signal $i \in N$, if $r_{ii} = 0$, then the bidder with a high valuation knows that the opponent has a low valuation, and therefore, both will bid 0. If $x_{1.i} = x_{2.i} = 0$ and $r_{ii} > 0$, then it is common knowledge that the valuations of both bidders are equal to 1. It is a standard argument to show that the unique equilibrium involves both bidders submitting bids equal to 1.

Suppose that $x_{l.i} \geq x_{m.i}$, $x_{l.i} > 0$, and $r_{ii} > 0$. First, note that $F_{l.i}\left(\bar{b}_i\right) = F_{m.i}\left(\bar{b}_i\right) = 1$ is satisfied, and so, the distribution functions (1) and (2) are well defined. The expected payoff of bidder l with valuation $v_l = 1$ is given by

$$\left\{\frac{px_{l.i}}{px_{l.i} + (1-p)\,r_{ii}} + \frac{(1-p)\,r_{ii}}{px_{l.i} + (1-p)\,r_{ii}} F_{m.i}(b)\right\}(1-b)\,. \tag{28}$$

Substituting (2) into (28), we can verify that bidder l is indifferent among all bids in the interval $\left[0, \bar{b}_i\right]$, earning the expected payoff given in (3). Similarly, the expected payoff of bidder m with valuation $v_m = 1$ is

$$\left\{\frac{px_{m.i}}{px_{m.i} + (1-p)\,r_{ii}} + \frac{(1-p)\,r_{ii}}{px_{m.i} + (1-p)\,r_{ii}} F_{l.i}(b)\right\}(1-b)\,. \tag{29}$$

Substituting (1) into (29), we can verify that bidder m is also indifferent among all bids in the interval $\left[0, \bar{b}_i\right]$, earning the same expected payoff given in (3). Obviously, no bidder has incentives to bid above $\bar{b}_i$, while any bid below 0 would give a payoff of 0. Thus, we can conclude that (1) and (2) represent the equilibrium strategies of high-valuation bidders when they observe the public signal i.

To prove that this equilibrium is unique, we can argue as in the proof of Proposition 1 that each bidder of type i will submit a bid according to an atomless distribution function, except possibly at 0; ties occur with zero probability; the supports of both distribution functions coincide; the common support is connected with the lower limit being equal to 0. Then, the payoff of bidders 1 and 2 are given by (28) and (29), respectively. The common support also implies that the equilibrium payoffs of both bidders are the same and equal to the expression in (3). Equating (28) and (29) with (3) gives (1) and (2). Hence, the equilibrium is unique.

Proof of Theorem 2 We partition all signals into two sets, S and $N\backslash S$. The former contains all signals i such that $x_{1.i} > x_{2.i}$, while the latter contains all signals such that $x_{1.i} < x_{2.i}$. Signals i for whom $x_{1.i} = x_{2.i}$ holds are assigned arbitrarily as long as each set is non-empty.

Using the results of Proposition 2, the joint ex-ante payoff of bidders is $p(1-p)$ times the following expression:

$$\sum_{i\in S} x_{1.i} + \sum_{i\in N\setminus S} x_{2.i} \frac{px_{1.i} + (1-p)\,r_{ii}}{px_{2.i} + (1-p)\,r_{ii}} + \sum_{i\in N\setminus S} x_{2.i} + \sum_{i\in S} x_{1.i} \frac{px_{2.i} + (1-p)\,r_{ii}}{px_{1.i} + (1-p)\,r_{ii}}.$$

The first two terms represent the payoff of bidder 1, and the other two terms—the payoff of bidder 2. Using

$$\sum_{i\in S} x_{1.i} + \sum_{i\in N\setminus S} x_{1.i} = 1,$$
$$\sum_{i\in S} x_{2.i} + \sum_{i\in N\setminus S} x_{2.i} = 1,$$

we can rewrite the joint payoff as

$$2 - \sum_{i\in N\setminus S} x_{1.i} + \sum_{i\in N\setminus S} x_{2.i} \frac{px_{1.i} + (1-p)\,r_{ii}}{px_{2.i} + (1-p)\,r_{ii}} - \sum_{i\in S} x_{2.i} + \sum_{i\in S} x_{1.i} \frac{px_{2.i} + (1-p)\,r_{ii}}{px_{1.i} + (1-p)\,r_{ii}}. \tag{30}$$

The expression in (30) is increasing in $x_{1.i}$ for all $i \in S$, but decreasing for all $i \in N\setminus S$. Similarly, (30) is increasing in $x_{2.i}$ for all $i \in N\setminus S$, but decreasing for all $i \in S$. Therefore, $x_{1.i} = 0$ for all $i \in N\setminus S$ and $x_{2.i} = 0$ for all $i \in S$, and we simplify (30) to

$$2 + \sum_{i\in N\setminus S} x_{2.i} \frac{(1-p)\,r_{ii}}{px_{2.i} + (1-p)\,r_{ii}} + \sum_{i\in S} x_{1.i} \frac{(1-p)\,r_{ii}}{px_{1.i} + (1-p)\,r_{ii}}.$$

One can verify that

$$\frac{ab}{a+b} + \frac{cd}{c+d} \leq \frac{(a+c)\,(b+d)}{a+b+c+d}.$$

(This inequality can be rewritten as $(ad - bc)^2 \geq 0$.) Therefore,

$$\sum_{i\in N\setminus S} x_{2.i} \frac{(1-p)\,r_{ii}}{px_{2.i} + (1-p)\,r_{ii}} \leq \frac{(1-p)\sum_{i\in N\setminus S} r_{ii}}{p + (1-p)\sum_{i\in N\setminus S} r_{ii}},$$
$$\sum_{i\in S} x_{1.i} \frac{(1-p)\,r_{ii}}{px_{1.i} + (1-p)\,r_{ii}} \leq \frac{(1-p)\sum_{i\in S} r_{ii}}{p + (1-p)\sum_{i\in S} r_{ii}},$$

where we have additionally used the fact that $\sum_{i\in N\setminus S} x_{2.i} = \sum_{i\in S} x_{1.i} = 1$. Therefore, we can increase the joint payoff if we aggregate all signals $i \in S$ into a (new) signal 1 and all signals $i \in N\setminus S$ into a (new) signal 2 such that $\tilde{x}_{1.1} = 1$, $\tilde{x}_{1.2} = 0$, $\tilde{x}_{2.1} = 0$, $\tilde{x}_{2.2} = 1$, $\tilde{r}_{11} = \sum_{i\in S} r_{ii}$, and $\tilde{r}_{22} = \sum_{i\in N\setminus S} r_{ii} = 1 - \tilde{r}_{11}$. Given these signals 1 and 2, we can apply Proposition 2 to verify that the joint equilibrium payoff is indeed given by

$$2+\frac{(1-p)\,\tilde{r}_{11}}{p+(1-p)\,\tilde{r}_{11}}+\frac{(1-p)\,\tilde{r}_{22}}{p+(1-p)\,\tilde{r}_{22}}. \tag{31}$$

This completes the proof that it is sufficient that the public signal takes one of the two values in order to achieve the maximal joint payoff.

To find the optimal distribution of signals, it remains to maximize (31) subject to $\tilde{r}_{11}+\tilde{r}_{22}=1$. It follows that $\tilde{r}_{11}=\tilde{r}_{22}=0.5$. According to (3), a high-valuation bidder expects a payoff of $\frac{2p}{1+p}$ irrespective of the public signal that she observes. The ex-ante payoff of bidder is $p\,(1-p)\,\frac{2}{1+p}$.

Proof of Proposition 3 First, note that $F_i\left(\bar{b}_i\right)=1$ for all $i\in N$ is satisfied, and so the mixed strategies are well defined. Suppose that bidder 2 follows the strategy given in Proposition 3 and $y_1>0$. Consider bidder 1 of type $i\in N$. Her expected payoff when bidding $b\in\left[\bar{b}_{i-1},\bar{b}_i\right]$ is

$$\left\{\frac{px_i+(1-p)\,y_i\sum_{k=1}^{i-1}y_k}{px_i+(1-p)\,y_i}+\frac{(1-p)\,y_i^2}{px_i+(1-p)\,y_i}F_i\,(b)\right\}(1-b)\,.$$

Substituting $F_i\,(b)$ from (4) yields a positive constant

$$\frac{px_i+(1-p)\,y_i\sum_{k=1}^{i-1}y_k}{px_i+(1-p)\,y_i}\left(1-\bar{b}_{i-1}\right).$$

Therefore, bidder 1 is indeed indifferent between any bid in the interval $\left[\bar{b}_{i-1},\bar{b}_i\right]$.

Suppose now that bidder 1 of type i bids in an interval $\left[\bar{b}_{j-1},\bar{b}_j\right]$ for $j\neq i$. Her expected payoff is

$$\begin{aligned}
&\left\{\frac{px_i+(1-p)\,y_i\sum_{k=1}^{j-1}y_k}{px_i+(1-p)\,y_i}+\frac{(1-p)\,y_iy_j}{px_i+(1-p)\,y_i}F_j\,(b)\right\}(1-b)\\
&=\frac{px_i+(1-p)\,y_i\sum_{k=1}^{j-1}y_k}{px_i+(1-p)\,y_i}(1-b)\\
&\quad+\frac{(1-p)\,y_iy_j}{px_i+(1-p)\,y_i}\,\frac{px_j+(1-p)\,y_j\sum_{k=1}^{j-1}y_k}{(1-p)\,y_j^2}\left(b-\bar{b}_{j-1}\right)\\
&=\frac{b}{px_i+(1-p)\,y_i}\left\{\frac{y_i}{y_j}\left(px_j+(1-p)\,y_j\sum_{k=1}^{j-1}y_k\right)\right.
\end{aligned}$$

$$- \left(px_i + (1-p)\, y_i \sum_{k=1}^{j-1} y_k \right) \Bigg\} + \Phi$$

$$= \frac{pb}{px_i + (1-p)\, y_i} \left(\frac{y_i}{y_j} x_j - x_i \right) + \Phi,$$

where the rest of the terms that do not contain b are collected in the parameter Φ. Since $x_j/y_j > x_i/y_i$ for all $j < i$, it follows that the payoff of type i is increasing in b for $b < \bar{b}_{i-1}$, and therefore, bidder 1 of type i does not want to deviate by bidding below $\bar{b}_{i-1}$. Similarly, since $x_j/y_j < x_i/y_i$ for all $j > i$, it follows that the payoff of type i is decreasing in b for $b > \bar{b}_i$, and therefore, bidder 1 of type i does not want to deviate by bidding above $\bar{b}_i$ either. The same argument establishes that there is no equilibrium, in which the supports of equilibrium strategies are arranged in a different order. That is, for any two types i and j such that $i < j$, it must be the case that the support of type i's mixed strategy must lie to the left of the support of type j's mixed strategy.

It remains to prove that supports cannot overlap in an equilibrium. Suppose on the contrary that an interval $\left[\underline{b}, \overline{b}\right]$ belongs to the support of equilibrium mixed strategies of more than one type. Let the set of these types be denoted by S. The expected payoff of type $i \in S$ from bidding in the interval $\left[\underline{b}, \overline{b}\right]$ is

$$\pi_i = \frac{px_i + (1-p)\, y_i \sum_{k \in N} y_k \tilde{F}_k (b)}{px_i + (1-p)\, y_i} (1-b)\,,$$

where $\tilde{F}_k (b)$ denotes the distribution of bids for type k in this equilibrium. If we multiply both sides with

$$\frac{y_j}{px_j + (1-p)\, y_j}$$

where $j \in S \setminus \{i\}$, we obtain

$$\frac{y_j \pi_i}{px_j + (1-p)\, y_j} = \frac{px_i y_j + (1-p)\, y_i y_j \sum_{k \in N} y_k \tilde{F}_k (b)}{(px_i + (1-p)\, y_i) \left(px_j + (1-p)\, y_j\right)} (1-b)\,.$$

If we subtract the analogous expression, in which the roles of types i and j are reversed, from the above expression, we have that

$$\frac{y_j \pi_i}{px_j + (1-p)\, y_j} - \frac{y_i \pi_j}{px_i + (1-p)\, y_i} = \frac{px_i y_j - px_j y_i}{(px_i + (1-p)\, y_i) \left(px_j + (1-p)\, y_j\right)} (1-b)\,.$$

Given that $x_i y_j \neq x_j y_i$, the above expression is satisfied only for a single value of b. Therefore, the supports of mixed strategies of types i and j cannot overlap in the equilibrium. This completes the proof that the symmetric equilibrium, described in Proposition 3, is unique. If $y_1 = 0$, using the tie-breaking rule, the proof is basically the same.

References

Aumann, R.J.: Subjectivity and correlation in randomized strategies. J. Math. Econ. **1**(1), 67–96 (1974)

Āzacis, H.: Information Disclosure by a Seller in a Sequential First Price Auction. Cardiff Business School, Mimeo (2013)

Āzacis, H., Vida, P.: Collusive Communication Schemes in a First-Price Auction. Cardiff Economics Working Paper E2012/11, Cardiff Business School (2012)

Bergemann, D., Morris, S.: Correlated Equilibrium in Games with Incomplete Information. Discussion Paper 1822, Cowles Foundation (2011).

Bergemann, D., Vällimäki, J.: Information in mechanism design. In: Blundell, R., Newey, W.K., Persson, T. (eds.) Advances in Economics and Econometrics: Theory and Applications, Ninth World Congress, Chap. 5, vol. 1, pp. 186–221. Cambridge University Press, Cambridge (2006)

Caffarra, C., Kühn, K.-U.: The cost of simplistic rules for assessing information exchange: the Italian jet fuel decision. In: Bergman, M. (ed.) The Pros and Cons of Information Sharing, pp. 131–177. Swedish Competition Authority, Stockholm (2006)

Chakravarty, S., Kaplan, T.R.: Optimal allocation without transfer payments. Games Econ. Behav. **77**, 1–20 (2013)

Condorelli, D.: What money can't buy: efficient mechanism design with costly signals. Games Econ. Behav. **75**, 613–624 (2012)

Cotter, K.D.: Correlated equilibrium in games with type-dependent strategies. J. Econ. Theory **54**, 48–68 (1991)

Ding, W., Jeitschko, T.D., Wolfstetter, E.G.: Signal jamming in a sequential auction. Econ. Lett. **108**, 58–61 (2010)

Eliaz, K., Forges, F.: Disclosing Information to Interacting Agents. Tel Aviv University, Mimeo (2012)

Eliaz, K., Serrano, R.: Sending information to interactive receivers playing a generalized prisoners dilemma. Int. J. Game Theory (2013). doi:10.1007/s00182-013-0374-x

Fang, H., Morris, S.: Multidimensional private value auctions. J. Econ. Theory **126**, 1–30 (2006)

Forges, F.: An approach to communication equilibria. Econometrica **54**(6), 1375–1385 (1986)

Forges, F.: Five legitimate definitions of correlated equilibrium in games with incomplete information. Theory Decis. **35**(3), 277–310 (1993)

Forges, F.: Correlated equilibrium in games with incomplete information revisited. Theory Decis. **61**(4), 329–344 (2006)

Genesove, D., Mullin, W.P.: The Sugar Institute Learns to Organize Information Exchange. Working Paper 5981, NBER (1997)

Hartline, J.D., Roughgarden, T.: Optimal mechanism design and money burning. In: Proceedings of the 40th Annual ACM Symposium on Theory of Computing, STOC '08, pp. 75–84 (2008)

Kaplan, T.R.: Communication of preferences in contests for contracts. Econ. Theory **51**(2), 487–503 (2012)

Kaplan, T.R., Zamir, S.: The Strategic Use of Seller Information in Private-Value Auctions. Discussion Paper 221. Hebrew University, The Center for the Study of Rationality (2000)

Kim, J., Che, Y.-K.: Asymmetric information about rivals' types in standard auctions. Games Econ. Behav. **46**, 383–397 (2004)

Landsberger, M., Rubinstein, J., Wolfstetter, E., Zamir, S.: First-price auctions when the ranking of valuations is common knowledge. Rev. Econ. Des. **6**(3), 461–480 (2001)

Lopomo, G., Marx, L.M., Sun, P.: Bidder collusion at first-price auctions. Rev. Econ. Des. **15**(3), 177–211 (2011)

Marshall, R.C., Marx, L.M.: Bidder collusion. J. Econ. Theory **133**, 374–402 (2007)

Marshall, R.C., Marx, L.M.: The Economics of Collusion: Cartels and Bidding Rings. MIT Press, Cambridge, MA (2012)

Maskin, E.S., Riley, J.G.: Auction theory with private values. Am. Econ. Rev. **75**(2), 150–155 (1985)

Maskin, E.S., Riley, J.G.: Asymmetric auctions. Rev. Econ. Stud. **67**(3), 413–438 (2000a)

Maskin, E.S., Riley, J.G.: Equilibrium in sealed high bid auctions. Rev. Econ. Stud. **67**(3), 439–454 (2000b)

McAfee, R.P., McMillan, J.: Bidding rings. Am. Econ. Rev. **82**(3), 579–599 (1992)

Miralles, A.: Self-enforced collusion through comparative cheap talk in simultaneous auctions with entry. Econ. Theory **42**(3), 523–538 (2010)

Quint, D.: Looking smart versus playing dumb in common-value auctions. Econ. Theory **44**(3), 469–490 (2010)

Skreta, V.: On the informed seller problem: optimal information disclosure. Rev. Econ. Des. **15**(1), 1–36 (2011)

Vives, X.: Trade association disclosure rules, incentives to share information, and welfare. RAND J. Econ. **21**(3), 409–430 (1990)

Yoon, K.: Optimal mechanism design when both allocative and expenditure inefficiency matter. J. Math. Econ. **47**, 670–676 (2011)